

1. $\frac{1-\ln 2}{2}$

By polynomial long division, $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$, so $\int_0^1 \frac{x^3}{x^2+1} dx$
 $= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) \Big|_0^1 = \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1-\ln 2}{2}$.

2. 8

Newton's Method iteration is defined as $y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$. For
 $f(y) = \frac{1}{y^2} - 2$, this becomes $y_{n+1} = y_n \left(\frac{3}{2} - y_n^2 \right)$. With initial guess
 $y_0 = 1$, we find that $y_1 = \frac{1}{2}$, and $y_2 = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{4} \right) = \frac{5}{8} = \frac{5}{2^3}$, so the answer
 is $5+3=8$.

3. $\frac{\pi}{4}$

The area is $\int_0^\pi x(t)y'(t)dt = \int_0^\pi (\sin t)(t \cos t + \sin t)dt$
 $= \int_0^\pi (t \sin t \cos t + \sin^2 t)dt = \int_0^\pi \left(\frac{1}{2}t \sin 2t + \frac{1}{2} - \frac{1}{2}\cos 2t \right)dt$
 $= -\frac{1}{4}t \cos 2t - \frac{1}{8}\sin 2t + \frac{1}{2}t - \frac{1}{4}\sin 2t \Big|_0^\pi = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$.

4. $-\frac{3}{2}$

The line must have a negative slope, and the line through this point has
 equation $y = m(x-2) + 3$, so the y -intercept is $-2m+3$ while the x -
 intercept is $2 - \frac{3}{m}$. Therefore, the area is $A = \frac{1}{2}(-2m+3)\left(2 - \frac{3}{m}\right)$
 $= -2m + 6 - \frac{9}{2m}$, making $A' = -2 + \frac{9}{2m^2}$, and $A' = 0$ makes $m = \pm \frac{3}{2}$.
 Since m is negative, $m = -\frac{3}{2}$, and A' changes from negative to positive
 at that value, this slope is the answer.

5. $\frac{7}{2}$

$f(x) = 6x^5 - 35x^4 + 70x^3 - 60x^2 \Rightarrow f'(x) = 30x^4 - 140x^3 + 210x^2 - 120x$
 $\Rightarrow f''(x) = 120x^3 - 420x^2 + 420x - 120 = 120(x-1)(x-2)(2x-1)$, so
 the zeros of $f''(x)$ are 1, 2, and $\frac{1}{2}$. Since the multiplicity of each zero is
 odd, the signs of $f''(x)$ change at each of those values, making them all
 inflection points. Therefore, $1 + 2 + \frac{1}{2} = \frac{7}{2}$.

6. 1

The integrand is even, so $\int_{-\infty}^{\infty} (|x|^3 e^{-x^2})dx = 2\int_0^{\infty} (x^3 e^{-x^2})dx$, and using
 integration by parts, $2\int_0^{\infty} (x^3 e^{-x^2})dx = 2\left(-\frac{1}{2}x^2 e^{-x^2} \Big|_0^{\infty} + \int_0^{\infty} x e^{-x^2} dx\right)$

- $= -x^2 e^{-x^2} - e^{-x^2} \Big|_0^\infty = (0+0) - (0-1) = 1.$
7. $\sin 1$ $\sin(x - \cos x) = \sin x \cdot \cos(\cos x) - \cos x \cdot \sin(\cos x)$, so $I + J$
 $= \int_0^{\frac{\pi}{2}} (\cos(\cos x) \sin x) dx = \int_{-1}^0 \cos u du = \sin u \Big|_{-1}^0 = 0 - \sin(-1) = \sin 1.$
8. $\frac{\sqrt[3]{36\pi}}{3}$ Since volume is $V = \frac{4}{3}\pi r^3$ and surface area is $S = 4\pi r^2$, rearrangement
yields $S = \sqrt[3]{36\pi} \cdot V^{\frac{2}{3}}$, so $\frac{dS}{dt} \Big|_{V=8} = \frac{2}{3} \sqrt[3]{36\pi} \cdot \frac{1}{\sqrt[3]{V}} \Big|_{V=8} = \frac{\sqrt[3]{36\pi}}{3}.$
9. $\frac{10}{3}$ $\frac{dy}{dt} = y(10 - 3y)$ represents a logistic growth model with asymptotes at
 $y = 0$ and $y = \frac{10}{3}$. Regardless of the specific solution, it must be the
case that $\lim_{t \rightarrow \infty} f(t) = \frac{10}{3}.$
10. $\frac{e-1}{e}$ $\lim_{n \rightarrow \infty} \left(\frac{a_n}{n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \left(e^{-\frac{i}{n}} \right) \right)$ is a Riemann sum representing $\int_0^1 e^{-x} dx$, so
this limit is $-e^{-x} \Big|_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e} = \frac{e-1}{e}.$
11. 0 The integrand is an odd function, so do the symmetry of the x-values
about the y-axis, this integral has value 0.
12. $\frac{1}{6}$ For $-1 \leq x \leq 0$, $x\{x\}^2 = x(x+1)^2$, and for $0 \leq x \leq 1$, $x\{x\}^2 = x^3$.
Therefore,

$$\int_{-1}^1 (x\{x\}^2) dx = \int_{-1}^0 (x(x+1)^2) dx + \int_0^1 (x^3) dx = \frac{1}{3}x(x+1)^3 - \frac{1}{12}(x+1)^4 \Big|_{-1}^0$$

$$+ \frac{1}{4}x^4 \Big|_0^1 = -\frac{1}{12} + \frac{1}{4} = \frac{1}{6}.$$
13. $\frac{\sqrt{15}}{60}$ $(f \circ g \circ h \circ k)'(2) = (f' \circ g \circ h \circ k)(2) \cdot (g' \circ h \circ k)(2) \cdot (h' \circ k)(2) \cdot k'(2)$
 $= f'\left(\frac{1}{4}\right) \cdot g'\left(\frac{4}{\sqrt{17}}\right) \cdot h'(4) \cdot 4.$ Now, $f(x) = \sqrt{1-x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{1-x^2}}$
 $\Rightarrow f'\left(\frac{1}{4}\right) = -\frac{1}{\sqrt{15}}.$ Also, $g(x) = \frac{\sqrt{1-x^2}}{x} \Rightarrow g'(x) = \frac{-1}{x^2 \sqrt{1-x^2}}$
 $\Rightarrow g'\left(\frac{4}{\sqrt{17}}\right) = -\frac{17\sqrt{17}}{16}.$ Further, $h(x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow h'(x) = \frac{1}{(x^2+1)^{\frac{3}{2}}}$
 $\Rightarrow h'(4) = \frac{1}{17\sqrt{17}}.$ Therefore, $(f \circ g \circ h \circ k)'(2)$

$$= -\frac{1}{\sqrt{15}} \cdot -\frac{17\sqrt{17}}{16} \cdot \frac{1}{17\sqrt{17}} \cdot 4 = \frac{1}{4\sqrt{15}} = \frac{\sqrt{15}}{60}.$$

14. $\sqrt[4]{e}$

First, write $x^{-\ln x} = e^{-(\ln x)^2}$, then make the substitution $x = e^u$, $dx = e^u du$

to get $\int_0^\infty (x^{-\ln x}) dx = \int_{-\infty}^\infty e^{u-u^2} du = \int_{-\infty}^\infty e^{-\left(u-\frac{1}{2}\right)^2 + \frac{1}{4}} du = \sqrt[4]{e} \int_{-\infty}^\infty e^{-\left(u-\frac{1}{2}\right)^2} du$

$$\sqrt[4]{e} \int_{-\infty}^\infty e^{-v^2} dv, \text{ so } \frac{\int_0^\infty (x^{-\ln x}) dx}{\int_{-\infty}^\infty (e^{-x^2}) dx} = \frac{\sqrt[4]{e} \int_{-\infty}^\infty (e^{-v^2}) dv}{\int_{-\infty}^\infty (e^{-x^2}) dx} = \sqrt[4]{e}.$$

15. 10

Those four equations are vertical and horizontal lines that enclose a rectangular region with dimensions $5\alpha \times 6\alpha$, so the enclosed area is $30\alpha^2$. Therefore, the average value of this area is

$$\frac{1}{1-(-1)} \int_{-1}^1 30\alpha^2 d\alpha = 5\alpha^3 \Big|_{-1}^1 = 5 - (-5) = 10.$$

16. $\frac{61}{120}$

All order derivatives will be defined, and since $f(0) = f'(0) = 0$ and

$$f''(0) = 1 + (f(0))^2 = 1, \text{ the first nonzero coefficient is } \frac{f''(0)}{2!} = \frac{1}{2}.$$

Further, $f^{(3)}(x) = 2f(x)f'(x) \Rightarrow f^{(3)}(0) = 0$;

$$f^{(4)}(x) = 2f(x)f''(x) + 2(f'(x))^2$$

$$\Rightarrow f^{(4)}(0) = 0; \text{ and}$$

$$f^{(5)}(x) = 2f(x)f^{(3)}(x) + 6f'(x)f''(x) \Rightarrow f^{(5)}(0) = 0. \text{ However,}$$

$$f^{(6)}(x) = 2f(x)f^{(4)}(x) + 8f'(x)f^{(3)}(x) + 6(f''(x))^2$$

$$\Rightarrow f^{(6)}(0) = 6. \text{ Therefore, the next nonzero coefficient is } \frac{f^{(6)}(0)}{6!} = \frac{1}{120},$$

$$\text{and the sum of those two coefficients is } \frac{1}{2} + \frac{1}{120} = \frac{61}{120}.$$

17. $e^2 - 1$

Using the substitution $u = e^x$, $du = e^x dx$,

$$\int_{-\infty}^{\ln 2} e^{x+e^x} dx = \int_{-\infty}^{\ln 2} e^x e^{e^x} dx = \int_0^2 e^u du$$

$$= e^u \Big|_0^2 = e^2 - 1.$$

18. $y = 14x - 8$
(must be in
this format)

$f(1) = 6$ and $f'(x) = 9x^2 + 4x + 1 \Rightarrow f'(1) = 14$, so the tangent line has equation $y = 14(x - 1) + 6 \Rightarrow y = 14x - 8$.

19. $\frac{\sqrt[3]{6\pi^2}}{\pi}$

We know that the volume is $12 = V = \pi r^2 h \Rightarrow h = \frac{12}{\pi r^2}$. Therefore, the surface area is $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{24}{r} \Rightarrow S' = 4\pi r - \frac{24}{r^2}$, and $S' = 0$ when $4\pi r^3 = 24 \Rightarrow r = \sqrt[3]{\frac{6}{\pi}} = \frac{\sqrt[3]{6\pi^2}}{\pi}$. S' changes from negative to positive at that value, meaning that is the radius that minimizes the surface area, which is the material needed to make the can.

20. 39

The given line has slope $-\frac{5}{8}$, so the slope of the perpendicular line through the origin must be $\frac{8}{5}$, giving that line the equation $y = \frac{8}{5}x$.

This line intersects the given line when

$$5x + 8\left(\frac{8}{5}x\right) = 10 \Rightarrow \frac{89}{5}x = 10 \Rightarrow x = \frac{50}{89}, \text{ so the answer is } 89 - 50 = 39.$$

21. $\frac{\sqrt{3}}{2}$

Multiplying both sides of the given differential equation by θ' and then integrating gives $\frac{1}{2}(\theta')^2 = -\sin\theta + C$. Using the initial condition, we have

$$\frac{1}{2}(0)^2 = -\sin\frac{\pi}{3} + C \Rightarrow C = \frac{\sqrt{3}}{2}. \text{ Therefore, } \frac{1}{2}(f'(t))^2 + \sin(f(t)) = \frac{\sqrt{3}}{2} \text{ for all values, including for } t = 1.$$

22. $\frac{2}{23}$

$$\lim_{x \rightarrow 0} \left(\frac{2x}{3x + 4\sin(5x)} \right) = \lim_{x \rightarrow 0} \left(\frac{2}{3 + \frac{4\sin(5x)}{x}} \right) = \frac{2}{3 + 4(5)} = \frac{2}{23}$$

23. $\frac{1}{256}$

$$\text{Using integration by parts, } \int (x^7 (\ln x)^2) dx = \frac{1}{8} x^8 (\ln x)^2 - \frac{1}{4} \int (x^7 \ln x) dx,$$

and then using integration by parts on this second integral, we get

$$\int (x^7 \ln x) dx = \frac{1}{8} x^8 \ln x - \int \frac{1}{8} x^7 dx = \frac{1}{8} x^8 \ln x - \frac{1}{64} x^8. \text{ Therefore,}$$

$$\int (x^7 (\ln x)^2) dx = \frac{1}{8} x^8 (\ln x)^2 - \frac{1}{32} x^8 \ln x + \frac{1}{256} x^8. \text{ Further, even though}$$

0 cannot be plugged into these first two terms, using L'hôpital's Rule, we can evaluate as

$$\int_0^1 (x^7 (\ln x)^2) dx = \frac{1}{8} x^8 (\ln x)^2 - \frac{1}{32} x^8 \ln x + \frac{1}{256} x^8 \Big|_0^1 = \frac{1}{256}.$$

24. 2π

This is the quarter of the unit circle in quadrant I, so when revolved about the x-axis, this creates half of the unit sphere, so the surface area is $\frac{1}{2}(4\pi(1)^2) = 2\pi$.

25. $2026(\ln 2026)^2$ This limit represents the derivative of the function $f(x) = 2026^{2026^x}$ evaluated at $x = 0$. $f'(x) = 2026^{2026^x} \ln 2026 \cdot 2026^x \ln 2026$, so $f'(0) = 2026 \ln 2026 \cdot \ln 2026 = 2026(\ln 2026)^2$.