

ANSWERS:

1. $\sqrt{29}$

2. 260

3. 3

4. $\frac{22}{9}$

5. $\frac{\sqrt{15}}{5}$

6. 32

7. $\frac{25\pi}{4}$

8. 128

9. -10

10. $\frac{\sqrt{3}}{6}$

11. $\frac{1}{2^{2027}}$ or 2^{-2027}

12. 2352

13. $-\frac{80}{41} + \frac{100}{41}i$

14. 6

15. 1

16. $2i$

17. $3\sqrt{2}$

18. 0

19. $\frac{\sqrt{6}-\sqrt{2}}{8}$

20. 125

21. 216

22. 24

23. π

24. $\frac{1441}{729}$

25. 12

SOLUTIONS:

1. $2 \sin x + 5 \cos x = t \left(\frac{2}{t} \sin x + \frac{5}{t} \cos x \right)$. Notice that this looks like $\sin(\alpha + x) = \cos \alpha \sin x + \sin \alpha \cos x$. If we set up a triangle with angle α , it will have sides 2, 5, and $\sqrt{2^2 + 5^2} = \sqrt{29} = t$, which is the amplitude of the function.

2. $A = (-3)(-4) + 4 \cdot 2 = 20$; $B = \sqrt{10^2 + 24^2} = 26$; $\text{lcm}(20, 26) = 260$

3. $x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)(x^2 + 1)$ which has three real solutions.

4. $\sum \frac{k+2^k}{4^k} = \sum \frac{k}{4^k} + \sum \left(\frac{1}{2}\right)^k$. The first we can use the formula $\frac{a_0}{(1-r)^2} = \frac{\frac{1}{4}}{\left(1-\frac{1}{4}\right)^2} = \frac{4}{9}$ and the second can use the formula $\frac{a_0}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$. The final sum is then $\frac{4}{9} + \frac{18}{9} = 22/9$.

5. $2x^2 + 8x + 5y^2 - 60y = 12 \Rightarrow \frac{(x+2)^2}{10^2} + \frac{(y-6)^2}{\sqrt{40}^2} = 1$. The eccentricity of an ellipse is $\sqrt{1 - \frac{b^2}{a^2}}$ where b is the minor radius and a is the major radius. We have $\sqrt{1 - \frac{40}{100}} = \sqrt{\frac{60}{100}} = \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5}$.

6. The value of x that we'd need to plug into $2x - 1$ to obtain $x + 5$ is $\frac{x+6}{2}$. We will plug this into the RHS of the given f , then find the product of the solutions: $f(x + 5) = 2 \left(\frac{x+6}{2}\right)^2 - 3 \left(\frac{x+6}{2}\right) + 7 = \frac{1}{2}x^2 + \frac{9}{2}x + 16$. The product is the last term divided by the first term, which is $\frac{16}{1/2} = 32$.

7. $r = a \sin \theta$ is a circle tangent to the origin with diameter a , meaning the radius is $\frac{a}{2}$ and the area for $a = 5$ is $\pi \left(\frac{5}{2}\right)^2 = \frac{25\pi}{4}$.

8. $\det(AB) = \det(A) \det(B)$. The determinant of each matrix can be calculated in a variety of ways, but I'll use cofactor reduction. The first matrix can be calculated along the bottom

row as $\det \begin{pmatrix} -2 & 4 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0 \cdot \det \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} -2 & 4 \\ 3 & 1 \end{pmatrix} = 2$,

and the second matrix can be done along the first row as $\det \begin{pmatrix} 4 & 2 & 0 \\ 1 & 6 & 7 \\ 3 & 0 & 1 \end{pmatrix} = 4 \cdot$

$\det \begin{pmatrix} 6 & 7 \\ 0 & 1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 1 & 7 \\ 3 & 1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 1 & 6 \\ 3 & 0 \end{pmatrix} = 64$. Finally, we have $2 \cdot 64 = 128$.

9. We need to get the polynomial from $\det \left(\begin{bmatrix} 6 & 8 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right) = 0 \Rightarrow (6-x)(-3-x) - (-1)(8) = 0 \Rightarrow x^2 - 3x - 10 = 0$, the product of the solutions is -10 .

10. Write the denominator as a difference of squares $\lim_{t \rightarrow 3} \frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} = \lim_{t \rightarrow 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{\sqrt{3}}{6}$.

11. We can repeatedly apply the double-angle formula for sine: $\frac{1}{2} \sin 2x = \sin x \cos x$, absorbing all powers of 2 and ending with $y = \frac{1}{2^{2027}} \sin(2^{2027}x)$.

12. We need to choose 2 boys from a group of 8, and 3 girls from a group of 9, which means we have $\binom{8}{2} \cdot \binom{9}{3} = \frac{8 \cdot 7}{2 \cdot 1} \cdot \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 2352$.

13. Fun fact: $(1+2i)^2 = -3+4i$ and $12+16i = 4(3+4i)$. We then have two cancellations giving us $\frac{(1+2i)}{1} \cdot \frac{(1-2i)}{1} \cdot \frac{4}{(-4-5i)} = \frac{20}{41}(-4+5i)$.

14. Only the last digit matters here, let's list out some powers of 4: $4^1 = 4, 4^2 = 16, 4^3 = 64, 4^4 = 256, 4^5 = 1024, 4^6 = 4096$ and so on. So if the exponent is even, it ends in a 6, and 14^{2026} is clearly even.

15. Let's do some transformations but stop short of actually solving. Using log properties, we have $\log_2 \left(\frac{(1+\sqrt{x})(\sqrt{x+1})}{2x} \right) = 3 \Rightarrow (1+\sqrt{x})(\sqrt{x+1}) = 16x$. The RHS is a line with a slope of 16 that passes thru the origin. For the LHS, we can approximate it by $(1+\sqrt{x})(\sqrt{x+1}) = \sqrt{x} + \sqrt{x^2+x} \approx x + \sqrt{x}$. Setting these two equal to each other we have $16x = x + \sqrt{x} \Rightarrow 15x = \sqrt{x}$ which has one solution in the first quadrant.

16. $i^2 \cdot (-i)^{-2} \cdot (1+i)^2 = (-1)(-1)(2i) = 2i$

17. The equation factors to $(x-2)(x^4+2x^3+4x^2+8x+16) = 0$, where the second factor has no real solutions. The product of these non-real solutions is $\frac{16}{1}$ and the sum is $-\frac{2}{1}$, so we ask for $\sqrt{16-(-2)} = \sqrt{18} = 3\sqrt{2}$.

18. The remainder when a polynomial is divided by a linear term like $x-a$ is exactly $f(a)$. In this case, we plug in $x=2$, and note that the second factor becomes 0, making the entire product 0.

19. We apply the sine double-angle formula again, noting that $\sin\left(\frac{\pi}{24}\right) \cos\left(\frac{\pi}{24}\right) = \frac{1}{2} \sin\left(\frac{\pi}{12}\right) = \frac{1}{2} \left(\frac{\sqrt{6}-\sqrt{2}}{4} \right)$. This can be found with the half-angle formula $\sin \frac{\theta}{2} = \sqrt{\frac{1-\cos \theta}{2}}$, or memorized.

20. We find the number of factors of 5 in 2026: $\left\lfloor \frac{2026}{5} \right\rfloor + \left\lfloor \frac{2026}{25} \right\rfloor + \left\lfloor \frac{2026}{125} \right\rfloor + \left\lfloor \frac{2026}{625} \right\rfloor = 405 + 81 + 16 + 3 = 505$. We now do the same for 505: $\left\lfloor \frac{505}{5} \right\rfloor + \left\lfloor \frac{505}{25} \right\rfloor + \left\lfloor \frac{505}{125} \right\rfloor = 101 + 20 + 4 = 125$.

21. We'll need to compute $\frac{4!}{i!j!k!} (x^2)^i (2x)^j \left(-\frac{3}{x}\right)^k$ with $i + j + k = 4$. For the constant term, the exponent of x will be 0, so we also need $2i + j - k = 0 \Rightarrow k = 2i + j$. Let's plug this back in to the first sum, giving us $i + j + (2i + j) = 3i + 2j = 4$. Note that i, j, k must be non-negative integers, so we'll do some casework. If $i = 0$, we have $2j = 4 \Rightarrow j = 2$ and $k = 2$. If $i = 1$, then $2j = 1$ which has no integer solution. If $i = 2$, we have $2j = -2$ which has no non-negative solutions. Therefore the only ordered triple is $(0, 2, 2)$ and the quantity we need is $\frac{4!}{0!2!2!} (x^2)^0 (2x)^2 \left(-\frac{3}{x}\right)^2 = 216$.

22. For the product of two periodic functions, the total period is the lcm of the periods of each factor. Tangent has a base period of π , and the modified version has a period of $\frac{\pi}{\pi/3} =$

3. Sine has a base period of 2π , and the modified period is $\frac{2\pi}{\pi/4} = 8$. Then we have

$$\text{lcm}(3, 8) = 24.$$

23. Let's note that $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$. Let's first find $\tan(\arctan 1 + \arctan 2) = \frac{\tan(\arctan 1) + \tan(\arctan 2)}{1 - \tan(\arctan 1) \tan(\arctan 2)} = \frac{1+2}{1-1 \cdot 2} = -3$. If we take the arctan of the far left and far right sides of the equality, we have $\arctan 1 + \arctan 2 = \arctan(-3)$, but note that since $\arctan 1 = \frac{\pi}{4}$ and $\frac{\pi}{4} < \arctan 2 < \frac{\pi}{2}$, we have $\arctan 1 + \arctan 2 \in \left(\frac{\pi}{2}, \pi\right)$. This must be rectified, as arctan is traditionally restricted to quadrants 1 and 4, but the LHS is in quadrant 2. We can use the fact that $\tan \theta = \tan(\theta + \pi)$, and the fact that tan (and arctan) are odd functions to write $\arctan 1 + \arctan 2 = \pi + \arctan(-3) = \pi - \arctan 3$. Let's add in the arctan 3 to get $\arctan 1 + \arctan 2 + \arctan 3 = \pi - \arctan 3 + \arctan 3 = \pi$.

24. Factor out a $\frac{1}{9}$ and get $\frac{1}{9} \sum_{k=0}^4 \left(\frac{5}{3}\right)^k$. We can use the geometric series formula $S =$

$$\frac{a_0(r^n - 1)}{r - 1} \Rightarrow \frac{1}{9} \cdot \frac{1\left(\left(\frac{5}{3}\right)^5 - 1\right)}{\frac{5}{3} - 1} = \frac{1}{9} \cdot \frac{\frac{3125}{243} - \frac{243}{243}}{\frac{2}{3}} = \frac{1}{9} \cdot \frac{2882}{243} \cdot \frac{3}{2} = \frac{1441}{729}.$$

25. $3^{18} \equiv 1 \pmod{19}$, so $3^{51} \equiv 3^{15} \pmod{19}$. So in $\pmod{19}$, $3^{15} = 27^5 \equiv 8^5 = 64^2 \cdot 8 \equiv 7^2 \cdot 8 = 49 \cdot 8 \equiv 11 \cdot 8 = 88 \equiv 12$, so the remainder is 12.