

1. C

2. A

3. D

4. B

5. B

6. B

7. C

8. D

9. A

10.C

11.A

12.C

13.B

14.A

15.D

16.A

17.B

18.E

19.B

20.B

21.A

22.B

23.D

24.C

25.C

26.A

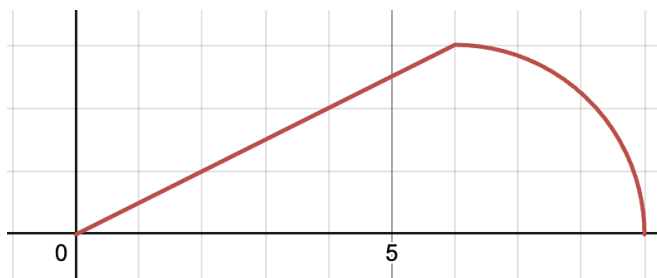
27.C

28.B

29.B

30.B

- C:** No matter the placement of point E , the triangle has the same height and base as the rectangle. So its area of the triangle is $\frac{1}{2}$ the area of the rectangle.
- A:** Let s be the side length of the cube. Then $6s^2 = 2\pi(4\sqrt{\pi})^2 + 2\pi(4\sqrt{\pi})(2\sqrt{\pi}) = 48\pi^2$. Therefore, $s^2 = 8\pi^2$ so $s^3 = 8^{\frac{3}{2}}\pi^3 = (2\sqrt{2})^3\pi^3 = 16\pi^3\sqrt{2}$.
- D:** Solve $2026x = 2026 - 2026x$ to get $x = \frac{1}{2}$. When $x = \frac{1}{2}$, the height of the triangle is $y = 2026\left(\frac{1}{2}\right) = 1013$. Solve $2026 - 2026x = 0$ to find $x = 1$, so the base of the triangle is 1. Therefore, the area of the triangle is $\frac{1}{2}(1)(1013) = \frac{1013}{2}$.
- B:** Let E be the foot of the perpendicular from C to $\overleftrightarrow{BD}$ and let $CE = h$, the height of the cone. Let $AE = x$. Then $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{h}{6+x}$ and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3} = \frac{h}{x}$. Therefore, $\frac{6+x}{\sqrt{3}} = \sqrt{3}x$ giving $x = 3$ and $h = 3\sqrt{3}$. The volume of the cone is $\frac{1}{3}\pi(3)^2(3\sqrt{3}) = 9\pi\sqrt{3}$.
- B:** Let $m\angle AMO = \theta$. The area of $\triangle MAO$ is $A(\theta) = \frac{1}{2}(8)(6)\sin(\theta) = 24\sin(\theta)$. Then $A'(\theta) = 24\cos(\theta) = 0$ when $\theta = \frac{\pi}{2}$. Since A' changes from positive to negative at $\theta = \frac{\pi}{2}$, then A is maximized for that angle. Then $A\left(\frac{\pi}{2}\right) = 24\sin\left(\frac{\pi}{2}\right) = 24$. More simply, the area is maximized for a right triangle.
- B:** By the ratio of the legs, we know the triangle is a 30-60-90 triangle. The side lengths are in the extended ratio $x:x\sqrt{3}:2x$ whose area is $x(3 + \sqrt{3}) = 12 + 4\sqrt{3}$, so $x = 4$. The sides lengths are then $4, 4\sqrt{3}, 8$ so the area is $\frac{1}{2}(4 \cdot 4\sqrt{3}) = 8\sqrt{3}$.
- C:** Complete the square to write $\sqrt{-x^2 + 12x - 27} = \sqrt{9 - (x - 6)^2}$. A quick sketch shows the function consists of a line through the origin and a quarter circle. When revolved, the solid is composed of a cone with radius 3 and height 6 with a hemisphere of radius 3. The total volume is $\frac{1}{3}\pi(3)^2(6) + \frac{2}{3}\pi(3)^3 = 18\pi + 18\pi = 36\pi$.



8. **D:** The area in question is $\int_0^\infty x^4 e^{-2x} dx = \lim_{b \rightarrow \infty} \int_0^b x^4 e^{-2x} dx$. Using integration by parts (or the tabular method):

Sign	D	I
+	x^4	e^{-2x}
−	$4x^3$	$-\frac{1}{2}e^{-2x}$
+	$12x^2$	$\frac{1}{4}e^{-2x}$
−	$24x$	$-\frac{1}{8}e^{-2x}$
+	24	$\frac{1}{16}e^{-2x}$
−	0	$-\frac{1}{32}e^{-2x}$

Therefore, $\int x^4 e^{-2x} dx = -e^{-2x} \left(\frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{3}{4} \right)$. As $x \rightarrow \infty$, the antiderivative function goes to zero due to an exponential function growing faster in the denominator. Thus, $\int_0^\infty x^4 e^{-2x} dx = 0 - \left(-\frac{3}{4} \right) = \frac{3}{4}$.

9. **A:** Let x be the length of the smaller base, $3x$ the length of the larger base, and h the height of the trapezoid. It is given that $3 \frac{dx}{dt} = -6$ so $\frac{dx}{dt} = -2$. The area then is $40 = \frac{1}{2}(x + 3x)h = 2xh$ so $20 = xh$. When $x = 2$, $h = 10$. Now derive with respect to time: $0 = \frac{dx}{dt}h + x \frac{dh}{dt}$. Substitute to find $0 = -2(10) + 2 \frac{dh}{dt}$ so $\frac{dh}{dt} = 10$. This means, the height is increasing at a rate of 10 units per minute.
10. **C:** Solve $x^2 + 2x = 12 - x^2$ to find $x = -3, 2$. The total area is $\int_{-3}^2 ((12 - x^2) - (x^2 + 2x)) dx$. Rewrite as $\int_{-3}^2 (-2x^2 - 2x + 12) dx$. Integrate and substitute to get: $\left(-\frac{2}{3}(2)^3 - (2)^2 + 12(2) \right) - \left(-\frac{2}{3}(-3)^3 - (-3)^2 + 12(-3) \right) = \frac{125}{3}$. Then $\int_{-3}^2 (k - x^2 - 2x) dx = \frac{125}{6}$. So $\left(2k - \frac{1}{3}(2)^3 - (2)^2 \right) - \left(-3k - \frac{1}{3}(-3)^3 - (-3)^2 \right) = \frac{125}{6}$. Solve $5k - \frac{8}{3} - 4 = \frac{125}{6}$ to get $k = \frac{11}{2}$. Alternatively, consider a graph. The maximum of $y = 12 - x^2$ is $y = 12$ and the minimum of $y = x^2 + 2x = (x + 1)^2 - 1$ is $y = -1$. The average of these extremes is $\frac{12-1}{2} = \frac{11}{2}$. Notice that the region in question has rotation symmetry about the center of the region, at point $\left(-\frac{1}{2}, \frac{11}{2} \right)$. Thus, $y = \frac{11}{2}$ will divide the region into two equal areas.



- 11. A:** Set $x^e = e^x$. By inspection, $x = e$. Using the washer method, the outer radius to the line $y = -e$ is $R = e^x + e$ and the inner radius is $r = x^e + e$. Thus, the volume is $\pi \int_0^e (R^2 - r^2) dx = \pi \int_0^e ((e^x + e)^2 - (x^e + e)^2) dx$.
- 12. C:** Letting $A = \pi r^2$, then $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. When $A = 25\pi$, then $r = 5$. At this instant, $\frac{dA}{dt} = 2\pi(5) \cdot \frac{5}{\pi} = 50$ square centimeters per second.
- 13. B:** The area of the rectangle is $A(x) = xe^{-2x}$. Then $A'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1 - 2x) = 0$ when $x = \frac{1}{2}$. Note $A'(x)$ changes from positive to negative at $x = \frac{1}{2}$, showing that $A(x)$ is maximum at $x = \frac{1}{2}$. The maximum area is $A\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1} = \frac{1}{2e}$.
- 14. A:** Note $f(1) = 0$. By even symmetry, the desired area is $2 \int_0^1 \frac{1-x^2}{\sqrt{x^2+1}} dx$. Let $x = \tan \theta$ so $\frac{dx}{d\theta} = \sec^2 \theta$. The new integral is $2 \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 \theta}{\sqrt{\tan^2 \theta + 1}} \sec^2 \theta d\theta$. Rewrite as $2 \int_0^{\frac{\pi}{4}} (2 - \sec^2 \theta) \sec \theta d\theta$. This gives $4 \int_0^{\frac{\pi}{4}} \sec \theta d\theta - 2 \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta$. The first evaluates to $4 \ln |\sec \frac{\pi}{4} + \tan \frac{\pi}{4}| = 4 \ln(\sqrt{2} + 1)$. For the second, use integration by parts with $u = \sec \theta$ and $dv = \sec^2 \theta d\theta$. We have:
 $I = \int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta$. Let $\tan^2 \theta = \sec^2 \theta - 1$ to get:
 $I = \int \sec^3 \theta d\theta = \sec \theta \tan \theta - I + \int \sec \theta d\theta$. So $I = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$.
Hence, $2 \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = \sec \frac{\pi}{4} \tan \frac{\pi}{4} + \ln \left(\sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right) = \sqrt{2} + \ln(\sqrt{2} + 1)$.
In total, we get $4 \ln(\sqrt{2} + 1) - (\sqrt{2} + \ln(\sqrt{2} + 1)) = 3 \ln(\sqrt{2} + 1) - \sqrt{2}$.
- 15. D:** Rewrite the equation as $r = 2\sqrt{2} + 2\sqrt{2} \left(\cos \theta \cdot \frac{\sqrt{2}}{2} - \sin \theta \cdot \frac{\sqrt{2}}{2} \right)$ or $r = 2\sqrt{2} \left(1 + \cos \left(\theta + \frac{\pi}{4} \right) \right)$. This represents a rotated cardioid, which has the same area as $r = 2\sqrt{2}(1 + \cos \theta)$. Its area is $\frac{1}{2} \int_0^{2\pi} \left(2\sqrt{2}(1 + \cos \theta) \right)^2 d\theta$. Square and simplify to get $r = 4 \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta$. Integrate separately:
 $4 \int_0^{2\pi} 1 d\theta = 4(2\pi) = 8\pi$
 $4 \int_0^{2\pi} 2 \cos \theta d\theta = 0$ since the integral is over a full period.
 $4 \int_0^{2\pi} \cos^2 \theta d\theta = 4 \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = 2(2\pi) + 0 = 4\pi$.
Summing gives $8\pi + 4\pi = 12\pi$.

16. A: Let the radius of T be r and the radius of S as $\frac{1}{2}r$. Then $\frac{V(S)}{V(T)} = \frac{\frac{4}{3}\pi\left(\frac{1}{2}r\right)^3}{\frac{4}{3}\pi r^3} = \frac{1}{8}$ or 1:8.

17. B: The side length is $s = 2\sqrt{\ln(x+1)}$. The area of a regular n -gon is $A = \frac{ns^2}{4\tan\left(\frac{\pi}{n}\right)} =$

$n \ln(x+1) \cot\left(\frac{\pi}{n}\right)$. For $n = 6$, the volume is $6\sqrt{3} \int_0^3 \ln(x+1) dx$. Use integration by parts on $\int \ln(x+1) dx$ with $u = \ln(x+1)$ and $dv = dx$ to find the antiderivative

$\int \ln(x+1) dx = x \ln(x+1) - \int \frac{x}{x+1} dx = x \ln(x+1) - \int 1 - \frac{1}{x+1} dx$. This becomes

$x \ln(x+1) - x + \ln(x+1) = \ln(x+1)(x+1) - x$. Then $\int_0^3 \ln(x+1) dx =$

$4 \ln(4) - 3$. Therefore, the volume is $6\sqrt{3}(4 \ln 4 - 3) = 24\sqrt{3} \ln 4 - 18\sqrt{3}$.

18. E: Since $2n$ is even, there are $4n$ petals. One petal spans the interval $\left[0, \frac{\pi}{2n}\right]$ since $r = 0$

at these endpoints. The area of one petal is $\frac{R^2}{2} \int_0^{\frac{\pi}{2n}} \sin^2(2n\theta) d\theta = \frac{R^2}{4} \int_0^{\frac{\pi}{2n}} (1 -$

$\cos(2n\theta)) d\theta$. Integrate and substitute the bounds to find $\frac{R^2}{4} \left(\frac{\pi}{2n} - \frac{\sin\left(2n\left(\frac{\pi}{2n}\right)\right)}{2n} \right) - 0 =$

$\frac{\pi R^2}{8n}$. Since there are $4n$ petals of equal area, the total interior area is $4n \left(\frac{\pi R^2}{8n} \right) = \frac{\pi R^2}{2}$.

Notice this area is independent of the number of petals!

19. B: The volume in a spherical cap is given by $V = \pi \left(Rh^2 - \frac{h^3}{3} \right) = 4\pi h^2 - \frac{\pi h^3}{3}$ since

$R = 4$. Then $\frac{dV}{dt} = 8\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}$. Substitute to find $4\pi = 12\pi \frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{1}{3}$ inches per second.

20. B: Solve for x^2 to get $x^2 = -n \ln y$. Note that $f(0) = 1$. Using the disk method, the volume is $\pi \int_0^1 x^2 dy = \lim_{b \rightarrow 0^+} \pi \int_b^1 -n \ln y dy = \lim_{b \rightarrow 0^+} -\pi n \int_b^1 \ln y dy$. Using integration

by parts, it can be shown that $\int \ln y dy = y \ln y - y$, therefore, the volume is

$\lim_{b \rightarrow 0^+} -\pi n(-1) - (b \ln b - b) = \pi n$. Thus, $\pi n = 2026\pi$ so $n = 2026$. Note that

$\lim_{b \rightarrow 0^+} b \ln b$ can be written as $\lim_{b \rightarrow 0^+} \frac{\ln b}{\frac{1}{b}} = \lim_{b \rightarrow 0^+} \frac{\frac{1}{b}}{-\frac{1}{b^2}} = 0$.

21. A: The surface area, when revolving about the y -axis, is given by $\int 2\pi x ds$ and

$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$. Note that for $r = \cos^2 \theta$, $\frac{dr}{d\theta} = -2 \sin \theta \cos \theta$. Then

$ds = \sqrt{\cos^4 \theta + 4 \sin^2 \theta \cos^2 \theta} = \cos \theta \sqrt{\cos^2 \theta + 4 \sin^2 \theta}$. Also, $x = r \cos \theta$, or $x = \cos^2 \theta \cdot \cos \theta = \cos^3 \theta$. The surface area is then $2\pi \int_0^{\frac{\pi}{2}} \cos^3 \theta \cdot \cos \theta \sqrt{\cos^2 \theta + 4 \sin^2 \theta} d\theta$.

This simplifies to $2\pi \int_0^{\frac{\pi}{2}} \cos^4 \theta \sqrt{\cos^2 \theta + 4 \sin^2 \theta}$.

22. **B:** $L = 2(\sqrt{2}) + 3(\sqrt{3}) + 3(2\sqrt{2}) + 2(4\sqrt{3}) = 8\sqrt{2} + 11\sqrt{3}$. Since f is increasing, the left approximation is an underestimate of K .

23. **D:** The Trapezoidal Rule error bound is $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \leq K$ for $a \leq x \leq b$. For $f(x) = \sin(2x)$, compute $f'(x) = 2 \cos(2x)$ and $f''(x) = -4 \sin(2x)$. On $\left[0, \frac{\pi}{4}\right]$,

$|f''(x)|$ is bounded by $y = 4$ so $K = 4$. The error bound is $\frac{4\left(\frac{\pi}{4}-0\right)^3}{12 \cdot 4^2} = \frac{4\pi^3}{64} \cdot \frac{1}{12 \cdot 4^2} = \frac{\pi^3}{3072}$.

24. **C:** The surface area is given by $2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$. For the given curve,

$x'(t) = 1, y'(t) = \frac{t}{\sqrt{1+t^2}}$. The area is $2\pi \int_0^1 \sqrt{1+t^2} \sqrt{1 + \frac{t^2}{1+t^2}} dt$, which can be written

as $2\pi \int_0^1 \sqrt{1+t^2} \sqrt{\frac{1+2t^2}{1+t^2}} dt = 2\pi \int_0^1 \sqrt{1+2t^2} dt$. Let $t = \frac{1}{\sqrt{2}} \tan \theta$ where

$\frac{dt}{d\theta} = \frac{1}{\sqrt{2}} \sec^2 \theta$. The new integral is $\sqrt{2}\pi \int_0^{\arctan(\sqrt{2})} \sec^3 \theta d\theta$. From #14, we already

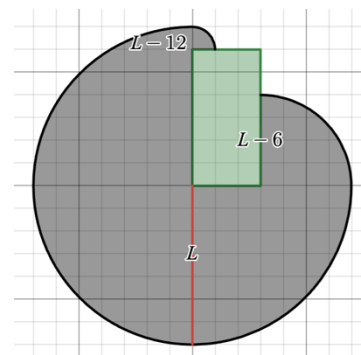
established $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$. Note that

$\sec(\arctan(\sqrt{2})) = \sqrt{3}$ by drawing a right triangle. The area is then

$\frac{\sqrt{2}}{2}\pi(\sqrt{3} \cdot \sqrt{2} + \ln(\sqrt{3} + \sqrt{2})) = \pi\sqrt{3} + \frac{\pi}{2}\sqrt{2}\ln(\sqrt{2} + \sqrt{3})$. Thus, $p + q + r + s = 3 + 2 + 2 + 3 = 10$.

25. **C:** Sketch a diagram of where the dog can travel. The dog is able to visit three possible regions: Three-quarters of the interior of a circle of radius L , a quarter circle of radius $L - 6$, and a quarter circle of radius $L - 12$. The total area is then

$\frac{3\pi}{4}L^2 + \frac{\pi}{4}(L-6)^2 + \frac{\pi}{4}(L-12)^2 = \frac{\pi}{4}(5L^2 - 36L + 180) = 164\pi$. This gives the quadratic $5L^2 - 36L - 476 = 0$. This factors as $(5L + 34)(L - 14) = 0$ so $L = 14$ since $L > 0$.



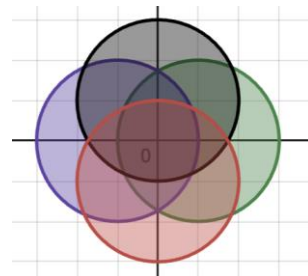
26. **A:** Consider a rectangular prism slice of volume of height h from the base of the trough and of thickness Δh . Then $\Delta V = wL\Delta h$. The force required to move the slice to the top is $1000gwL\Delta h$. The distance the slice needs to move to the top is $H - h$. The work required is then $1000gwL(H - h)\Delta h$. Note by similar triangles, $\frac{w}{A} = \frac{h}{H}$ so $w = \frac{A}{H}h$. Thus,

the work required is $\frac{1000gAL}{H}h(H-h)\Delta h$, where $0 \leq h \leq H$. Therefore, the total work required is $\frac{1000gAL}{H} \int_0^H h(H-h)dh = \frac{500gALH^2}{3}$.

- 27. C:** By symmetry, consider the portion of the solid in the first octant with intercepts $(a-1, 0)$, $(0, \frac{1}{a^2})$, $(a-1, 0)$. The base area is $\frac{1}{2}(a-1) \cdot \frac{1}{a^2} = \frac{a-1}{2a^2}$. The volume in the first octant is then $\frac{1}{3} \left(\frac{a-1}{2a^2} \right) (a-1) = \frac{(a-1)^2}{6a^2}$. The total volume is $8 \cdot \frac{(a-1)^2}{6a^2} = \frac{4(a-1)^2}{3a^2}$. Since the degrees of the numerator and denominator are both 2, the limit is $\frac{4}{3}$.

- 28. B:** The area of R is $A(k) = \int_0^{2k} \frac{x}{\sqrt{x^2+16}} dx$. Then $\frac{dA}{dt} = \frac{2k}{\sqrt{(2k)^2+16}} \cdot 2 \cdot \frac{dk}{dt}$ using the Fundamental Theorem of Calculus and the Chain Rule. Substitute $\frac{dA}{dt} = \frac{18}{5}$, $\frac{dk}{dt} = 3$, simplify to get $\frac{3}{5} = \frac{k}{\sqrt{k^2+4}}$. Square to get $\frac{9}{25} = \frac{k^2}{k^2+4}$ or $9k^2 + 36 = 25k^2$. Therefore, $36 = 16k^2$ so $k = \sqrt{\frac{36}{16}} = \frac{3}{2}$.

- 29. B:** Consider the boundaries of the four regions. Write as $(x-1)^2 + y^2 = 4$, $(x+1)^2 + y^2 = 4$, $x^2 + (y-1)^2 = 4$, and $x^2 + (y+1)^2 = 4$. The region $R \cap S \cap T \cap U$ is the intersection of the four disks contained by these circles. Since there is symmetry about $y=0$, $x=0$, and $y=x$, consider $\frac{1}{8}$ of the targeted region bounded by $y=0$, $(x+1)^2 + y^2 = 4$, and $y=x$. Convert to polar. Then $x^2 + 2x + y^2 = 3$ so $r^2 + 2r \cos \theta - 3 = 0$. Then $r = -\cos \theta + \sqrt{\cos^2 \theta + 3}$ and $r^2 = 3 + 2\cos \theta \sqrt{\cos^2 \theta + 3}$. Now consider the area given by $\frac{1}{2} \int_0^{\frac{\pi}{4}} (3 + 2\cos^2 \theta - 2\cos \theta \sqrt{\cos^2 \theta + 3}) d\theta$.



$$\frac{1}{2} \int_0^{\frac{\pi}{4}} 3 d\theta = \frac{3}{2} \cdot \frac{\pi}{4} = \frac{3\pi}{8}.$$

$$\frac{1}{2} \int_0^{\frac{\pi}{4}} 2 \cos^2 \theta d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta) d\theta = \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_0^{\frac{\pi}{4}} = \frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{\pi}{8} + \frac{1}{4}.$$

$$-\frac{1}{2} \int_0^{\frac{\pi}{4}} 2 \cos \theta \sqrt{\cos^2 \theta + 3} d\theta = -\int_0^{\frac{\pi}{4}} \cos \theta \sqrt{4 - \sin^2 \theta} d\theta. \text{ Let } u = \sin \theta. \text{ The integral}$$

becomes $-\int_0^{\frac{\sqrt{2}}{2}} \sqrt{4-u^2} du$. Let $u = 2 \sin t$ so $\frac{du}{dt} = 2 \cos t$. Rewrite as

$$-4 \int_0^{\arcsin(\frac{\sqrt{2}}{4})} \cos^2 t dt \text{ or } -2 \int_0^{\arcsin(\frac{\sqrt{2}}{4})} (1 + \cos 2t) dt = -2 \left(t + \frac{\sin 2t}{2} \right) \Big|_0^{\arcsin(\frac{\sqrt{2}}{4})} =$$

$$-2 \left(t + \sin t \cos t \right) \Big|_0^{\arcsin(\frac{\sqrt{2}}{4})}. \text{ This evaluates to } -2 \arcsin\left(\frac{\sqrt{2}}{4}\right) - \frac{\sqrt{7}}{4}.$$

Adding the results gives $\frac{\pi}{2} + \frac{1}{4} - 2 \arcsin\left(\frac{\sqrt{2}}{4}\right) - \frac{\sqrt{7}}{4}$. Multiply by 8 to get $4\pi + 2 - 16 \arcsin\left(\frac{\sqrt{2}}{4}\right) - 2\sqrt{7}$.

30. B: By symmetry, the optimal rectangle is a square with side length $\frac{16\sqrt{2}}{4} = 4\sqrt{2}$. Its area is $(4\sqrt{2})^2 = 32$. Alternatively, let $2w + 2h = 16\sqrt{2}$ so $h = 8\sqrt{2} - w$. The area is then $A(w) = -w^2 + 8\sqrt{2}w$. The maximum occurs at the vertex at $w = -\frac{8\sqrt{2}}{2(-1)} = 4\sqrt{2}$. Thus, $h = 4\sqrt{2}$ as well as the area is $(4\sqrt{2})^2 = 32$ as before.