

Answer Key:

1. A
2. B
3. A
4. C
5. D

6. D
7. C
8. A
9. A
10. D

11. D
12. E
13. B
14. C
15. C

16. D
17. A
18. B
19. D
20. C

21. C
22. D
23. B
24. C
25. B

26. B
27. A
28. B
29. D
30. A

Solutions:

1. A: Splitting into two fractions gives $\lim_{x \rightarrow \infty} \left(\frac{\cos(x)}{x} - 1 \right)$. Noting that $-1 \leq \cos(x) \leq 1$ for all x , we can bound the first term by $-\frac{1}{x} \leq \frac{\cos(x)}{x} \leq \frac{1}{x}$, and since the terms on both side limit to 0 as $x \rightarrow \infty$, by the squeeze theorem the first term is 0, and thus the total limit is $0 - 1 = \boxed{-1}$.

2. B: Rewrite the first improper integral as $\lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow 0^+} [2\sqrt{x}]_b^1 = 2$, so the integral converges. Rewrite the second improper integral as $\lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} [2\sqrt{x}]_1^b$, which approaches infinity, so the integral diverges. Thus the answer is yes; no.

3. A: The integral states that these two expressions either both converge or both diverge. We consider the latter. Substituting $u = \ln(x)$ gives $\int_2^\infty \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_2^\infty = \frac{1}{2}$, so the integral converges. Thus the answer is yes; yes.

4. C: The slope of the tangent line is given by $\frac{dy/dt}{dx/dt} = \frac{\sin(t)}{-\cos(t)} = \boxed{-\tan(t)}$.

5. D: Integrate by parts with $u = \ln(x)$, $dv = x^{2016} dx$ (and thus $du = \frac{1}{x} dx$, $v = \frac{1}{2017} x^{2017}$) to get

$$\left[\frac{1}{2017} x^{2017} \ln(x) \right]_0^1 - \int_0^1 \frac{1}{2017} x^{2016} dx = 0 - \left[\frac{1}{2017^2} x^{2017} \right]_0^1 = \boxed{-\frac{1}{2017^2}}.$$

6. D: Factor the denominator as $(x+1)(x+3)$, and we set up partial fractions:

$$\frac{1}{x^2 + 4x + 3} = \frac{A}{x+1} + \frac{B}{x+3}$$

Clearing denominators gives $1 = A(x+3) + B(x+1) = (A+B)x + (3A+B)$. It follows that $A+B=0$ and $3A+B=1$, from which it follows that $A = \frac{1}{2}$, $B = -\frac{1}{2}$. We thus want to integrate.

$$\int_1^2 \left(\frac{1/2}{x+1} - \frac{1/2}{x+3} \right) dx = \left[\frac{1}{2} \ln(x+1) - \frac{1}{2} \ln(x+3) \right]_1^2 = \left[\frac{1}{2} \ln \left(\frac{x+1}{x+3} \right) \right]_1^2 = \frac{1}{2} \left(\ln \left(\frac{3}{5} \right) - \ln \left(\frac{2}{4} \right) \right) = \boxed{\frac{1}{2} \ln \left(\frac{6}{5} \right)}.$$

7. C: This time the denominator has no real roots, so we complete the square instead to get $(x-2)^2 + 1$. Taking $u = x-2$, the integral becomes

$$\int_{-1}^0 \frac{1}{u^2 + 1} du = [\arctan(u)]_{-1}^0 = \boxed{\frac{\pi}{4}}.$$

8. A: Under the given definition, a cusp is a point where $x(t)$ and $y(t)$ change directions at the same time. A necessary condition for this is that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ need to both be 0 or undefined for the same value of t . We compute $x'(t) = 2t-2$ and $y'(t) = 3t^2-3$. $x'(t) = 0$ only when $t = 1$, and conveniently $y'(1) = 0$ as well. Furthermore, both $x'(t)$ and $y'(t)$ switch from negative to positive at $t = 1$, so $t = 1$ is our desired cusp.

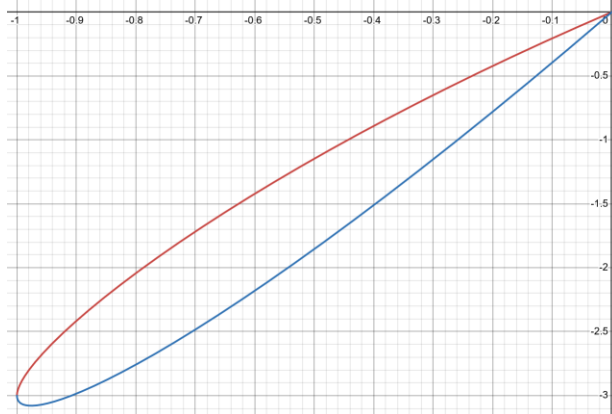
Of course, we still need to plug it in, getting $x(1) = -1$ and $x(2) = -2$, for a sum of $\boxed{-3}$.

9. A: We want two distinct values t_1, t_2 such that $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. Plugging in gives $t_1^2 - 2t_1 = t_2^2 - 2t_2$ and $t_1^3 - 4t_1 = t_2^3 - 4t_2$.

1. Subtracting the two sides of the first equation and factoring gives $(t_1 + t_2)(t_1 - t_2) - 2(t_1 - t_2) = 0 \rightarrow (t_1 + t_2 - 2)(t_1 - t_2) = 0$. Since $t_1 \neq t_2$ by assumption, this means $t_1 + t_2 = 2$.
2. Subtracting the two sides of the second equation and factoring gives $(t_1 - t_2)(t_1^2 + t_1t_2 + t_2^2) - 4(t_1 - t_2) = 0 \rightarrow (t_1 - t_2)(t_1^2 + t_1t_2 + t_2^2 - 4) = 0$. It follows that $t_1^2 + t_1t_2 + t_2^2 = 4$.
3. Plugging in $t_2 = 2 - t_1$ from the previous equation into this gives $t_1^2 + t_1(2 - t_1) + (2 - t_1)^2 = 4 \rightarrow t_1^2 - 2t_1 = 0$, so $t_1 = 0$ or 2 and $t_2 = 2$ or 0 .

In either case, the point of self-intersection is when $t = 0$ and $t = 2$, and the enclosed loop is traced when $0 \leq t \leq 2$.

Now we need to find the area. Here's a graph of the curve over the interval of interest (the red portion is when $0 \leq t \leq 1$, and the blue portion is when $1 \leq t \leq 2$; one can infer the general shape of the graph by looking at how $x(t)$ and $y(t)$ vary over the interval. In particular the fact that $x(t)$ hits a minimum at $t = 1$).



Now we integrate! We'll find the areas bounded between the x -axis and the curve for each of $0 \leq t \leq 1$ and $1 \leq t \leq 2$, and then subtract to get the area enclosed by the curve. We start with the former:

$$\int_0^1 y(t) dx(t) = \int_0^1 (t^3 - 4t)(2t - 2) dt = \int_0^1 (2t^4 - 2t^3 - 8t^2 + 8t) dt = \left[\frac{2}{5}t^5 - \frac{1}{2}t^4 - \frac{8}{3}t^3 + 4t^2 \right]_0^1 = \frac{37}{30}$$

Fortunately we don't have to integrate again and can just plug in for $1 \leq x \leq 2$.

$$\left[\frac{2}{5}t^5 - \frac{1}{2}t^4 - \frac{8}{3}t^3 + 4t^2 \right]_1^2 = -\frac{53}{30}$$

The areas in question are the absolute values of these (the sign differences are the result of tracing the curve in different directions and the curve being below the x -axis). Thus our area is $\frac{53}{30} - \frac{37}{30} = \boxed{\frac{8}{15}}$.

10. D: The slope of the tangent line is given by $\frac{dy/d\theta}{dx/d\theta} = \frac{d((2+\cos(\theta))(\sin(\theta)))/d\theta}{d((2+\cos(\theta))(\cos(\theta)))/d\theta} = \frac{2\cos(\theta) + \cos^2(\theta) - \sin^2(\theta)}{-2\sin(\theta) - 2\cos(\theta)\sin(\theta)}$.

Plugging in $\theta = \frac{\pi}{6}$ gives $\frac{\sqrt{3} + \frac{3}{4} - \frac{1}{4}}{-1 - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)} = \frac{2\sqrt{3} + 1}{-2 - \sqrt{3}} = \left(\frac{2\sqrt{3} + 1}{-2 - \sqrt{3}}\right) \left(\frac{-2 + \sqrt{3}}{-2 + \sqrt{3}}\right) = \boxed{4 - 3\sqrt{3}}$.

11. D: The area is given by $\frac{1}{2} \int_0^{2\pi} (2 + \cos(\theta))^2 d\theta = \frac{1}{2} \int_0^{2\pi} 4 + 4\cos(\theta) + \cos^2(\theta) d\theta$. The first two terms integrate to $8\pi + 0$, so it remains to evaluate $\frac{1}{2} \int_0^{2\pi} \cos^2(\theta) d\theta$. Doing the usual double angle trick gives $\frac{1}{2} \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} d\theta = \frac{\pi}{2}$, so the total integral is $\boxed{\frac{9\pi}{2}}$.

12. E: The ratio or root test will work here, let's use the root test. We have $\sqrt[n]{\frac{x^{2n}}{4^n n^2}} = \frac{x^2}{4\sqrt[n]{n^2}}$. Letting $L = \lim_{n \rightarrow \infty} \sqrt[n]{n^2}$, we have $\ln(L) = \lim_{n \rightarrow \infty} \frac{2 \ln(n)}{n} = 0$, so $L = 1$. The root test thus implies the series converges when $\frac{x^2}{4} < 1 \leftrightarrow x \in (-2, 2)$ and diverges when $\frac{x^2}{4} > 1 \leftrightarrow x \in (-\infty, -2) \cup (2, \infty)$.

The test is inconclusive when $x = \pm 2$, so we have to check those manually. Doing so gives the series $\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2}$. Both of these converge absolutely by the p -series test, so the interval of convergence is $\boxed{[-2, 2]}$.

13. B: We proceed by partial fractions. We want

$$\frac{1}{n^2 - 1} = \frac{A}{n + 1} + \frac{B}{n - 1}$$

Clearing denominators gives $1 = A(n - 1) + B(n + 1)$. Matching coefficients gives us $A + B = 0$ and $-A + B = 1$, so $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. We then evaluate starting at $n = 4$:

$$\left(\frac{1/2}{3} - \frac{1/2}{5}\right) + \left(\frac{1/2}{4} - \frac{1/2}{6}\right) + \left(\frac{1/2}{5} - \frac{1/2}{7}\right) + \dots$$

Notice that every term but $\frac{1/2}{3}$ and $\frac{1/2}{4}$ cancels with a later term, so the sum telescopes to simply $\frac{1}{6} + \frac{1}{8} = \boxed{\frac{7}{24}}$.

14. C: We proceed by partial fractions. We want

$$\frac{x^2 + x + 1}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

Clearing denominators gives $x^2 + x + 1 = (Ax + B)(x^2 + 1) + Cx + D$. Matching x^3 and x^2 coefficients give us $A = 0$ and $B = 1$, while subsequently matching x and 1 coefficients gives $C = 1$ and $D = 0$. Thus we want to integrate

$$\int_0^1 \frac{1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} dx = [\arctan(x)]_0^1 + \frac{1}{2} \int_0^1 \frac{1}{(u + 1)^2} du = \frac{\pi}{4} + \frac{1}{2} \left[-\frac{1}{u + 1} \right]_0^1 = \boxed{\frac{\pi + 1}{4}}.$$

15. C: Note that this region is symmetric about the y -axis, so we consider the portion where $y \geq 0$ and multiply the resulting area by 2. For $y \geq 0$ we can rearrange the equation to $y = \sqrt{1 - x^4}$. From here we use the shell method on the interval $x \in [0, 1]$.

$$\int_0^1 2\pi x \sqrt{1 - x^4} dx = 2\pi \left(\frac{1}{2}\right) \int_0^1 \sqrt{1 - x^2} dx.$$

The latter integral is a quarter circle with radius 1 and area $\frac{\pi}{4}$, so the total integral is $\frac{\pi^2}{4}$. Remembering to

multiply by 2 from earlier gives $\boxed{\frac{\pi^2}{2}}$.

16. D: Substitute $u = \sin(t)$, $du = \cos(t) dt$ into $\int_0^x e^{\sin^{-1}(t)} dt$ to get $\int_0^{\sin^{-1}(x)} e^u \cos(u) du$. Letting this integral be I , Integrating by parts twice (differentiating the trig term and integrating e^u) gives

$$\begin{aligned} I &= [e^u \cos(u)]_0^{\sin^{-1}(x)} - \int_0^{\sin^{-1}(x)} (-e^u \sin(u)) du \\ &= (e^{\sin^{-1}(x)} \sqrt{1-x^2} - 1) + [e^u \sin(u)]_0^{\sin^{-1}(x)} - \int_0^{\sin^{-1}(x)} e^u \cos(u) du \\ &= e^{\sin^{-1}(x)} (\sqrt{1-x^2} + x) - 1 - I, \end{aligned}$$

So $I = \frac{1}{2}(e^{\sin^{-1}(x)}(\sqrt{1-x^2} + x) - 1)$ and thus $f(x) = \frac{1}{2}(\sqrt{1-x^2} + x)$ while $C = -\frac{1}{2}$. Then we have

$$\frac{1}{2} \int_0^1 \sqrt{1-x^2} + \frac{1}{2} \int_0^1 x = \frac{\pi+2}{8}.$$

Adding $C = -\frac{1}{2}$ gives $\boxed{\frac{\pi-2}{8}}$.

17. A: We draw inspiration from the method to integrate $\cos^2(\theta)$ (turning it into $\frac{1+\cos(2\theta)}{2}$) and the fact that we can integrate $\int_0^\infty e^{-x} \cos(ax) dx$ by parts for any a . We first derive a triple angle formula for cosine:

$$\begin{aligned} \cos(3\theta) &= \cos(\theta + 2\theta) = \cos(\theta)(2\cos^2(\theta) - 1) - \sin(\theta)(2\sin(\theta)\cos(\theta)) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta)(1 - \cos^2(\theta)) = 4\cos^3(\theta) - 3\cos(\theta) \end{aligned}$$

We can rearrange this to get $\cos^3(\theta) = \frac{\cos(3\theta) + 3\cos(\theta)}{4}$, so that's the first part. Now we integrate $e^{-x} \cos(ax)$ by the exact same method as in the previous problem to get $\frac{e^{-x}(a \sin(ax) - \cos(ax))}{a^2 + 1}$. Taking the bounds 0 and ∞ gives the final integral a value of $\frac{1}{a^2 + 1}$. Combining this with the first part gives

$$\int_0^\infty e^{-x} \cos^3(x) dx = \int_0^\infty \frac{1}{4} e^{-x} \cos(3x) + \frac{3}{4} e^{-x} \cos(x) dx = \frac{1}{4} \left(\frac{1}{3^2 + 1} \right) + \frac{3}{4} \left(\frac{1}{1^2 + 1} \right) = \boxed{\frac{2}{5}}.$$

18. B: We compute the first three derivatives of $\sqrt{x}$ to be $\frac{1}{2}x^{-\frac{1}{2}}$, $-\frac{1}{4}x^{-\frac{3}{2}}$, and $\frac{3}{8}x^{-\frac{5}{2}}$, and evaluating at 1 gives $\frac{1}{2}$, $-\frac{1}{4}$, and $\frac{3}{8}$ respectively. This gives a Taylor expansion of $\frac{1}{0!} + \frac{1/2}{1!} + \frac{-1/4}{2!} + \frac{3/8}{3!} = \boxed{\frac{23}{16}}$

19. D: Integrate by parts with $u = \tan^{-1}(x)$, $dv = \frac{1}{(1+x)^2} dx$ to get

$$\left[-\frac{\tan^{-1}(x)}{1+x} \right]_0^1 - \int_0^1 \frac{1}{x^2+1} \left(-\frac{1}{x+1} \right) dx.$$

The former term is $-\frac{\pi}{4}$, and we use partial fractions on the latter integral. Write $\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$ and clear denominators to get $(Ax+B)(x+1) + C(x^2+1)$. Matching coefficients gives $A+C=0$, $A+B=0$, and $B+C=1$. Solving gives $A=-\frac{1}{2}$, $B=C=\frac{1}{2}$. Thus we want to integrate

$$\begin{aligned} \int_0^1 \frac{-\frac{1}{2}x+1}{x^2+1} + \frac{\frac{1}{2}}{x+1} dx &= \int_0^1 \frac{-\frac{1}{4}(2x)}{x^2+1} + \frac{1}{x^2+1} + \frac{\frac{1}{2}}{x+1} dx \\ &= \left[-\frac{1}{4} \ln(x^2+1) + \arctan(x) + \frac{1}{2} \ln(x+1) \right]_0^1 \\ &= \frac{\ln(2)}{4} + \frac{\pi}{4}. \end{aligned}$$

Adding $-\frac{\pi}{4}$ from earlier gives $\boxed{\frac{\ln(2)}{4}}$

20. C: We use the formula for surface area $\int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx$. Plugging in gives

$$2\pi \int_0^6 \sqrt{x + \frac{1}{4}} dx = \frac{4\pi}{3} \left[\left(x + \frac{1}{4}\right)^{\frac{3}{2}} \right]_0^6 = \boxed{\frac{62\pi}{3}}.$$

21. C: We use the formula for polar arc length $\int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2} d\theta$. Plugging in gives

$$\begin{aligned} \int_0^{2\pi} \sqrt{4 + 8 \cos(\theta) + 4 \cos^2(\theta) + 4 \sin^2(\theta)} d\theta &= \int_0^{2\pi} 4 \sqrt{\frac{1 + \cos(\theta)}{2}} d\theta \\ &= 4 \int_0^{2\pi} \left| \cos\left(\frac{\theta}{2}\right) \right| d\theta = 8 \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 8 \left[2 \sin\left(\frac{\theta}{2}\right) \right]_0^{\pi} = \boxed{16}. \end{aligned}$$

22. D: We'll let $x_0 = 0, \dots, x_4 = 2$ be our x -values, and $y_0 = 1, \dots, y_4$ be our resulting approximations.

y_1 : We compute $y' = 0 + 1 = 1$, so $y_1 = 1 + 1 \left(\frac{1}{2}\right) = \frac{3}{2}$.

y_2 : We compute $y' = \frac{1}{2} + \frac{3}{2} = 2$, so $y_2 = \frac{3}{2} + 2 \left(\frac{1}{2}\right) = \frac{5}{2}$.

y_3 : We compute $y' = 1 + \frac{5}{2} = \frac{7}{2}$, so $y_3 = \frac{5}{2} + \frac{7}{2} \left(\frac{1}{2}\right) = \frac{17}{4}$.

y_4 : We compute $y' = \frac{3}{2} + \frac{17}{4} = \frac{23}{4}$, so $y_4 = \frac{17}{4} + \frac{23}{4} \left(\frac{1}{2}\right) = \boxed{\frac{57}{8}}$.

23. B: Following the hint, we want $I(x)$ such that $y'I(x) - yI(x)$ looks like a product rule, which happens when $I(x) = -I'(x)$. Rearranging gives $\frac{I'(x)}{I(x)} = -1 \rightarrow \ln(I(x)) = -x + C \rightarrow I(x) = Ce^{-x}$. Picking $C = 1$ arbitrarily (anything nonzero will work), multiplying into the original equation gives us $[ye^{-x}]' = xe^{-x}$. Integrating both side (the RHS by parts) gives $ye^{-x} = -xe^{-x} - e^{-x} + C$. Plugging in $y(0) = 1$ gives $C = 2$, and multiplying through by e^{-x} gives $y = 2e^x - x - 1$. Plugging in $x = 2$ gives $\boxed{2e^2 - 3}$.

Our approximation was equal to 7.125 while the exact value is ~ 13.78 ... not the best approximation, but okay for an exponential function (errors compound for convex up functions).

24. C: Write $\sin^2(x) = 1 - \cos^2(x)$, and then the integral becomes $\int_0^{\pi/2} \cos^4(x) (1 - \cos^2(x)) \sin(x) dx$. Substituting $u = \cos(x)$, $du = -\sin(x) dx$ gives $-\int_1^0 (u^4 - u^6) du = \left[\frac{1}{5}u^5 - \frac{1}{7}u^7 \right]_0^1 = \boxed{\frac{2}{35}}$.

25. B: Let I be the original integral. Rewriting $\tan^2(x) = \sec^2(x) - 1$, we get the original integral is equal to $\int_0^{\pi/4} \sec^3(x) - \sec(x) dx$. The latter term equals $[-\ln(\sec(x) + \tan(x))]_0^{\pi/4} = -\ln(1 + \sqrt{2})$. We now integrate by parts on the first term with $u = \sec(x)$, $dv = \sec^2(x) dx$ to get

$$\int_0^{\pi/4} \sec^3(x) dx = [\sec(x) \tan(x)]_0^{\pi/4} - \int_0^{\pi/4} \sec(x) \tan^2(x) = \sqrt{2} - I$$

This means $I = -\ln(1 + \sqrt{2}) + \sqrt{2} - I$, or $I = \boxed{\frac{1}{2}(\sqrt{2} - \ln(1 + \sqrt{2}))}$.

26. B: Cancel out a factor of n to get $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{(n-1)!}$. We want to cancel out another factor, but we need an $n - 1$ rather than an n . So let's force one in! Rewrite as $\sum_{n=1}^{\infty} \frac{(n-1)2^n}{(n-1)!} + \frac{2^n}{(n-1)!}$. Split into $\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} + \sum_{n=1}^{\infty} \frac{2^n}{(n-1)!}$ and rewrite as $4 \sum_{n=0}^{\infty} \frac{2^n}{n!} + 2 \sum_{n=0}^{\infty} \frac{2^n}{n!}$. The sum term is equal to e^2 (by the Maclaurin series for e^x , so we get $\boxed{6e^2}$).

27. A: Noting the symmetry, it suffices to consider a single petal. We want to find the weighted average distance from a point in the petal to the origin. We can think of the weight of a given distance as the arc length of all points inside the petal at that distance.

For a given distance r , considering the petal at $\theta = 0$, we want $0 \leq r \leq \cos(2\theta)$, so $-\frac{1}{2}\arccos(r) \leq 0 \leq \frac{1}{2}\arccos(r)$, thus the arc length in question is $r \arccos(r)$. Thus to compute the weighted average, we want

$$\frac{\int_0^1 r^2 \arccos(r) dr}{\int_0^1 r \arccos(r) dr}.$$

For the first we substitute $r = \cos(x)$ to get

$$\int_0^{\pi/2} \cos^2(x) x \sin(x) dx = \left[-\frac{1}{3} x \cos^3(x) \right]_0^{\pi/2} - \int_0^{\pi/2} -\frac{1}{3} \cos^3(x) dx = \frac{2}{9},$$

Where the first equality follows from IBP and the second follows from the work in the next problem (I wrote that solution first).

For the second we substitute $r = \cos(x)$ again to get

$$\int_0^{\pi/2} \cos(x) x \sin(x) dx = \left[\frac{1}{2} x \sin^2(x) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2} \sin^2(x) dx = \frac{\pi}{4} - \int_0^{\pi/2} \frac{1 - \cos(2x)}{2} dx = \frac{\pi}{8}$$

Dividing the former by the latter gives $\boxed{\frac{16}{9\pi}}$.

28. B: Substitute $x = 3 \tan(\theta)$, $dx = 3 \sec^2(\theta) d\theta$ to get

$$\begin{aligned} \int_0^{\pi/2} \frac{3 \sec^2(\theta)}{(9 \tan^2(\theta) + 9)^{5/2}} dx &= \int_0^{\pi/2} \frac{\sec^2(\theta)}{81 \sec^5(\theta)} d\theta = \int_0^{\pi/2} \frac{1}{81} \cos^3(\theta) d\theta = \int_0^{\pi/2} \frac{1}{81} \cos(\theta) (1 - \sin^2(\theta)) d\theta \\ &= \frac{1}{81} \int_0^1 (1 - u^2) du = \frac{1}{81} \left[u - \frac{u^3}{3} \right]_0^1 = \boxed{\frac{2}{243}}. \end{aligned}$$

29. D:

I: Note that $n^2 + n + 1$ is always odd, so $(-1)^{n^2+n+1}$ is -1 , thus it suffices to determine if $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n+1}}$ converges. Limit comparison with $\frac{1}{n}$ shows this diverges.

II: We use the ratio test:

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n+1}{n+1} \cdot \frac{n^n}{(n+1)^n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \rightarrow \frac{1}{e} \text{ as } n \rightarrow \infty,$$

so the series converges.

III: Multiply by the conjugate to get $\frac{(n^2+2n+2)-(n^2+n+1)}{n\sqrt{n}(\sqrt{n^2+2n+2}+\sqrt{n^2+n+1})} = \frac{n+1}{n\sqrt{n}(\sqrt{n^2+2n+2}+\sqrt{n^2+n+1})}$. The third term is on the order of n , so limit comparison test with $\frac{1}{n\sqrt{n}}$ shows this converges.

IV: We use the root test:

$$\sqrt[n]{\frac{(2n)^{n/2}}{n^n}} = \frac{\sqrt{2n}}{n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

So the series converges.

In total, $\boxed{3}$ of these series converge.

30. A: We want $\int_0^k \sqrt{1 + f'(x)^2} dx = \ln(\sec(k) + \tan(k))$. Differentiating both sides, the first FTC gives $\sqrt{1 + f'(x)^2} = \sec(k)$, so $f'(x) = \tan(x)$ and thus $f(x) = \ln(\sec(x))$, which is increasing over the interval $\left[0, \frac{\pi}{2}\right)$.

Thus we want to evaluate $\int_0^{\pi/2} \ln(\sec(x)) dx = -\int_0^{\pi/2} \ln(\cos(x)) dx$. Substituting $u = \frac{\pi}{2} - x$ gives $-\int_{\frac{\pi}{2}}^0 -\ln(\sin(x)) dx = -\int_0^{\pi/2} -\ln(\sin(x)) dx$. Calling both of these I and adding gives

$$2I = \int_0^{\pi/2} -\ln(\sin(x) \cos(x)) dx = \int_0^{\pi/2} -\ln\left(\frac{1}{2} \sin(2x)\right) = \frac{\pi}{2} \ln(2) - \int_0^{\pi} \ln(\sin(u)) du = \frac{\pi}{2} \ln(2) + I$$

Thus $I = \boxed{\frac{\pi \ln(2)}{2}}$.