

Answers:

1. B
2. E
3. C
4. A
5. B
6. B
7. A
8. C
9. B
10. C
11. A
12. B
13. D
14. A
15. A
16. A
17. E
18. D
19. C
20. B
21. B
22. E
23. D
24. B
25. D
26. B
27. E
28. C
29. A
30. C

Solutions:

$$1. 2 = \frac{(-2k) - (-8)}{(k+1) - 3} = \frac{-2k+8}{k-2} \Rightarrow 2k-4 = -2k+8 \Rightarrow k=3$$

2. There is no x^4 term, so the sum of the solutions is 0.

3. Let S be the set of five-digit positive integers which are not multiples of 5 and consist only of the digits 1, 2, 3, 4, or 5. To find the number of elements of S , since the units digit can't be 5, there are 4 ways to assign the units digit, and $4!$ ways to assign the others. The number of elements of S less than 20,000 all have leading digit 1, and thus have 3 ways to assign the units digit and $3!$ ways to assign the rest. Hence the number of elements of S greater than 20,000 is $4 \cdot 4! - 3 \cdot 3! = 78$.

4. The number of ways of selecting b and x from the set is $7 \cdot 6 = 42$. However, $b \neq 1$, removing 6 ways, and if $x = 1$, all of the logarithms equal 0, so we remove 5 more ways to get down to 31. Finally, since $\log_2 4 = \log_3 9$, $\log_4 2 = \log_9 3$, $\log_2 3 = \log_4 9$, and $\log_3 2 = \log_9 4$, we remove the 4 ways to obtain equal values, leaving 27 distinct values.

5. If the leading digit is even, it can be 2, 4, 6, or 8, for 4 ways. The other two digits can be any of the odd digits 1, 3, 5, 7, or 9 for each of the remaining two digits. Hence, there are $4 \cdot 5 \cdot 5 = 100$ ways if the leading digit is even. If the middle digit is even, we can now use the digit 0, so there are $5 \cdot 5 \cdot 5 = 125$ ways. Likewise, there are 125 ways if the units digit is even, making a total of 350 different ways.

6. Let $S = \{n \in \mathbb{Z} \mid 1 \leq n \leq 2026\}$. The number of subsets T of the set $S \setminus \{2026\}$ is 2^{2025} . Now, we have a choice, either 2026 is in a subset T or it isn't. If the size of the subset is odd, we do not include 2026; if the size of the subset is even, we do include 2026. Hence we have only one choice for what to do with 2026. It follows that the number of subsets of S which have an odd number of elements is $2^{2025} \cdot 1 = 2^{2025}$.

7. Let k , $k+2$, and $k+4$ be the consecutive odd positive integers. Then k , $k+1$, $k+2$, $k+3$, $k+4$, and $k+5$ are six consecutive integers, meaning no number 7 or greater can divide any 2 of these. Further, two of these numbers are divisible by 3, one even and one odd, and only one of these numbers is divisible by 5. None of the odd integers are divisible by 2, and if the number divisible by 5 is even, we only have 3 as the common divisor.

8. The sum of the digits is clearly divisible by 9, and the sum of the digits of 111,111,111,111 is 12, so the number is not divisible by 81. The alternating digit sum is 0, so the number is divisible by 11; further, the sum $999 - 999 + 999 - 999 = 0$, so the number is divisible by 7 (and hence, 999,999,999,999 is divisible by 77). Dividing by 7, we get 999,999,999,999 $= 7 \cdot 142,857,142,857$, and $857 - 142 + 857 - 142$ is not divisible by 7, so 999,999,999,999 is not divisible by 49. Further, in modulo 17, $10^{12} - 1 = 100^6 - 1 \equiv (-2)^6 - 1 = 63 \equiv 12$, not 0, so 999,999,999,999 is not divisible by 17, hence not divisible by 51.

9. Let s be the sum of the fourteen labels. If we replace any -1 with 1, then there are four values in the sum, say a, b, c , and d (the vertex itself and the labels on its three adjacent faces), whose signs have changed, and the sum is now s' . Since $a + b + c + d$ is even, we see that $s - s' = 2(a + b + c + d)$ is a multiple of 4. Now repeat this process of replacing any -1 with 1 until all vertices are 1. Then the sum is $1 + 1 + \dots + 1 = 14 \equiv 2 \pmod{4}$. Hence, it is impossible for this sum to be 0.

$$\begin{aligned}
 10. \quad \tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ &= \tan 9^\circ - \tan 27^\circ - \cot 27^\circ + \cot 9^\circ = \frac{\sin 9^\circ}{\cos 9^\circ} + \frac{\cos 9^\circ}{\sin 9^\circ} \\
 &- \left(\frac{\sin 27^\circ}{\cos 27^\circ} + \frac{\cos 27^\circ}{\sin 27^\circ} \right) = \frac{\sin^2 9^\circ + \cos^2 9^\circ}{\sin 9^\circ \cos 9^\circ} - \frac{\sin^2 27^\circ + \cos^2 27^\circ}{\sin 27^\circ \cos 27^\circ} = \frac{1}{\sin 9^\circ \cos 9^\circ} - \frac{1}{\sin 27^\circ \cos 27^\circ} \\
 &= \frac{2}{2 \sin 9^\circ \cos 9^\circ} - \frac{2}{2 \sin 27^\circ \cos 27^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} = \frac{2(\sin 54^\circ - \sin 18^\circ)}{\sin 18^\circ \sin 54^\circ} = \frac{2(2 \cos 36^\circ \sin 18^\circ)}{\sin 18^\circ \sin 54^\circ} \\
 &= \frac{4(\sin 54^\circ \sin 18^\circ)}{\sin 18^\circ \sin 54^\circ} = 4
 \end{aligned}$$

11. The circumradius R is given by $R = \frac{abc}{4K}$, where a, b , and c are the side lengths of the triangle and K is the triangle's area. Using Hero's formula, the semi-perimeter is $s = \frac{13+14+15}{2} = 21$, so the area is $\sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21(8)(7)(6)} = 84$, so the circumradius has length $R = \frac{(13)(14)(15)}{4(84)} = \frac{65}{8}$, and the closest integer is 8.

12. We want $\frac{b}{a+1} > 2 \Rightarrow b > 2a + 2$. Translating this inequality to the xy -plane, we graph $y > 2x + 2$ within the square $0 \leq x, y \leq 10$. The line $y = 2x + 2$ intersects the left side of the square at $(0, 2)$ and the top of the square at $(4, 10)$, and the region where $y > 2x + 2$ is the

triangular region with vertices $(0,2)$, $(0,10)$, and $(4,10)$. The area of this region is

$$\frac{1}{2}(10-2)(4-0)=16, \text{ so the probability is } \frac{16}{10^2} = \frac{16}{100} = \frac{4}{25}.$$

13. Let $u=1+2\sqrt{x}$, then $x=\frac{1}{4}(u-1)^2$ and $dx=\frac{1}{2}(u-1)du$. Also, plugging into the first

substitution, $x=1 \Rightarrow u=3$ and $x=9 \Rightarrow u=7$. The integral then becomes $\int_3^7 \left(\frac{3}{u} \cdot \frac{1}{2}(u-1) \right) du$

$$= \frac{3}{2} \int_3^7 \left(1 - \frac{1}{u} \right) du = \frac{3}{2} (u - \ln|u|) \Big|_3^7 = \frac{3}{2} (7 - \ln 7 - 3 + \ln 3) = \frac{3}{2} \left(4 + \ln \frac{3}{7} \right) = 6 + \frac{3}{2} \ln \frac{3}{7}.$$

$$14. k = \int_1^2 (kx^2 - 9x + 1) dx = \left(\frac{k}{3}x^3 - \frac{9}{2}x^2 + x \right) \Big|_1^2 = \left(\frac{8}{3}k - 18 + 2 \right) - \left(\frac{1}{3}k - \frac{9}{2} + 1 \right) = \frac{7}{3}k - \frac{25}{2}$$

$$\Rightarrow \frac{4}{3}k = \frac{25}{2} \Rightarrow k = \frac{75}{8}$$

$$15. \left(\frac{\sec^3 x}{\tan^6 x} \right)^2 \left(\frac{\cos^2 x}{\cot^4 x} \right)^3 = \frac{\sec^6 x \cos^6 x}{\tan^{12} x \cot^{12} x} = \frac{1}{1} = 1$$

16. Substituting $25z$ for x yields $f(5z) = (25z)^2 - 10(25z) + 3 = 625z^2 - 250z + 3$, so

$$625z^2 - 250z + 3 = 2026 \Rightarrow 625z^2 - 250z - 2023 = 0, \text{ and the sum of the solutions is } -\frac{-250}{625} = \frac{2}{5}.$$

17. Since $n^4 + n^2 + 1 = (n^4 + 2n^2 + 1) - n^2 = (n^2 + 1)^2 - n^2 = (n^2 - n + 1)(n^2 + n + 1)$, we use partial

fractions to write $\frac{n}{n^4 + n^2 + 1} = \frac{1/2}{n^2 - n + 1} - \frac{1/2}{n^2 + n + 1}$, and thus the series telescopes:

$$\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1} \right) = \frac{1}{2} \left((1 - \cancel{1}) + \left(\cancel{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{13} \right) + \dots \right) = \frac{1}{2} (1) = \frac{1}{2}.$$

$$18. 3(\cos 123^\circ + i \sin 123^\circ) \cdot 4(\cos 282^\circ + i \sin 282^\circ) = (3 \cdot 4)(\cos(123^\circ + 282^\circ) + i \sin(123^\circ + 282^\circ))$$

$$12(\cos 405^\circ + i \sin 405^\circ) = 12 \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) = 6\sqrt{2} + 6\sqrt{2}i, \text{ so } |6\sqrt{2} \cdot 6\sqrt{2}| = 72.$$

19. Using the division algorithm for polynomials, write $P(x) = (x-3)(x-1)q(x) + (a+bx)$.

Then $1 = P(3) = a + 3b$ and $-1 = P(1) = a + b$, and solving this system yields $a = -2$ and $b = 1$, making $|b - a| = |1 - (-2)| = 3$.

20. $q(x) = (x+1)p(x) - 1$ is a degree 5 polynomial with roots 1, 2, 3, 4, and 5, so write it as

$q(x) = (x+1)p(x) - 1 = a(x-1)(x-2)(x-3)(x-4)(x-5)$. Plugging in $x = -1$ yields

$$-1 = a(-2)(-3)(-4)(-5)(-6) = -6!a \Rightarrow a = \frac{1}{6!}, \text{ which makes } (x+1)p(x) - 1$$

$$= \frac{1}{6!}(x-1)(x-2)(x-3)(x-4)(x-5). \text{ Now, plugging in } x = 6 \text{ yields } 7p(6) - 1$$

$$= \frac{1}{6!}(5)(4)(3)(2)(1) = \frac{1}{6} \Rightarrow 7p(6) = \frac{7}{6} \Rightarrow p(6) = \frac{1}{6}.$$

21. $2026 \equiv 1 \pmod{3}$, so $f(2026) = \lfloor \sqrt{2026} \rfloor = 45$. $45 \equiv 0 \pmod{3}$, so $f(45) = \lfloor \sqrt{45} \rfloor = 6$.

$6 \equiv 0 \pmod{3}$, so $f(6) = \lfloor \sqrt{6} \rfloor = 2$. $2 \equiv 2 \pmod{3}$, so $f(2) = \lfloor 2\sqrt{2} \rfloor + 1 = 3$. $3 \equiv 0 \pmod{3}$, so

$f(3) = \lfloor \sqrt{3} \rfloor = 1$. Therefore, it takes 5 copies of f to get to 1.

$$22. \log 1.3\overline{6} = \log \left(1\frac{4}{11} \right) = \log \frac{15}{11} = \log 3 + \log 5 - \log 11 = A + B - C$$

23. To be tangent to both axes, the center of the circle must have equal coordinates, so letting r be the radius length, the equation of the circle is $(x-r)^2 + (y-r)^2 = r^2$. Plugging in $(8,1)$ yields $r^2 = (8-r)^2 + (1-r)^2 = 64 - 16r + r^2 + 1 - 2r + r^2 \Rightarrow 0 = r^2 - 18r + 65 = (r-5)(r-13)$, so 13 is an option for the length of the radius (and 5 is the other).

$$24. z = \frac{1}{2}(\sqrt{2} + \sqrt{2}i + i\sqrt{2} - \sqrt{2}) = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}, \text{ so } z^{2026} = \cos \frac{2026\pi}{8} + i \sin \frac{2026\pi}{8} = \cos \frac{5\pi}{4}$$

$$+ i \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \text{ and } z^{14} = \cos \frac{14\pi}{8} + i \sin \frac{14\pi}{8} = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i. \text{ Therefore,}$$

$$w = z^{2026} + z^{14} + 1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + 1 = 1 - \sqrt{2}i, \text{ and } |w| = \sqrt{1^2 + (-\sqrt{2})^2} = \sqrt{1+2} = \sqrt{3}$$

≈ 1.732 .

25. Let the slope of the line be $-m$, then the equation of the line is $y = -m(x-2) + 5$. Its intercepts are $(0, 2m+5)$ and $(2 + \frac{5}{m}, 0)$, making the area enclosed by the triangle

$A = \frac{1}{2}(5+2m)\left(2 + \frac{5}{m}\right) = 10 + \frac{25}{2m} + 2m \Rightarrow A' = -\frac{25}{2m^2} + 2 \Rightarrow A' = 0$ when $m = \frac{5}{2}$, and sign analysis reveals this to be a minimum area. Therefore, the minimum enclosed area is

$$A = \frac{1}{2}\left(5 + 2 \cdot \frac{5}{2}\right)\left(2 + \frac{5}{\frac{5}{2}}\right) = \frac{1}{2}(10)(4) = 20.$$

26. Let $\angle ACB = \alpha$. It follows that $\angle CAD = \alpha$ since $\overline{BC}$ and $\overline{AD}$ are parallel. Now use the law of cosines on $\triangle ABC$ and $\triangle ACD$. We get $AB^2 = 25^2 + AC^2 - 50AC \cos \alpha$ and $CD^2 = 49^2 + AC^2 - 98AC \cos \alpha$, and thus since $AB^2 : CD^2 = 25 : 49$, we have

$$\frac{25^2 + AC^2 - 50AC \cos \alpha}{49^2 + AC^2 - 98AC \cos \alpha} = \frac{25}{49} \Rightarrow 49 \cdot 25^2 + 49AC^2 - 2450AC \cos \alpha = 25 \cdot 49^2 + 25AC^2 - 2450AC \cos \alpha$$

$$\Rightarrow 24AC^2 = 25 \cdot 49^2 - 49 \cdot 25 = 25 \cdot 49 \cdot 24 \Rightarrow AC^2 = 25 \cdot 49 \Rightarrow AC = 5 \cdot 7 = 35.$$

27. Using $\tan^2 \theta = \sec^2 \theta - 1$, $\tan^2 \theta + 4 \sec \theta + 2026 = \sec^2 \theta - 1 + 4 \sec \theta + 2026 = \sec^2 \theta + 4 \sec \theta + 2025 = (\sec \theta + 2)^2 + 2021$, so the minimum value is 2021, which occurs when $\sec \theta = -2$ ($\theta = \frac{2\pi}{3}$ is one such angle).

28. $a(t) = \langle 3, -1, 1 \rangle$, so $v(t) = \langle 3t + m, -t + n, t + p \rangle$. Since the direction of travel is $\langle 3, -1, 1 \rangle$, and the speed at $t = 0$ is 2, $\left\langle \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle = \frac{2\langle 3, -1, 1 \rangle}{\|\langle 3, -1, 1 \rangle\|} = v(0) = \langle m, n, p \rangle$, which means that

$$v(t) = \left\langle 3t + \frac{6}{\sqrt{11}}, -t - \frac{2}{\sqrt{11}}, t + \frac{2}{\sqrt{11}} \right\rangle. \text{ Therefore,}$$

$$p(t) = \left\langle \frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + b, -\frac{1}{2}t^2 - \frac{2}{\sqrt{11}}t + c, \frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + d \right\rangle \Rightarrow \langle 1, 2, 3 \rangle = p(0) = \langle b, c, d \rangle, \text{ which}$$

$$\text{means } p(t) = \left\langle \frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1, -\frac{1}{2}t^2 - \frac{2}{\sqrt{11}}t + 2, \frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3 \right\rangle. \text{ Therefore,}$$

$$p(2\sqrt{11}) = \langle 79, -24, 29 \rangle, \text{ so the particle's position at this time is } (79, -24, 29), \text{ and } 79 + (-24) + 29 = 84.$$

29. Let the centers of the three spheres be C_1 , C_2 , and C_3 , tangent to the plane at M , A , and T , respectively, and let the radii be $C_1M=r_1$, $C_2A=r_2$, and $C_3T=r_3$. Let P be the point on $\overline{C_2A}$ such that $\overline{C_1P}$ is parallel to $\overline{MA}$. Then $C_1P=MA$, $C_2P=|r_1-r_2|$, and $C_1C_2=r_1+r_2$. By the Pythagorean Theorem, $MA^2+(r_1-r_2)^2=(r_1+r_2)^2 \Rightarrow MA^2=4r_1r_2$. By similar reasoning, $AT^2=4r_2r_3$ and $TM^2=4r_1r_3$. Multiplying these equations together gives us $MA^2AT^2TM^2=64r_1^2r_2^2r_3^2 \Rightarrow r_1r_2r_3=\frac{1}{8}(MA \cdot AT \cdot TM)=\frac{1}{8}(18 \cdot 21 \cdot 24)=1134$ (the radii themselves are $r_1=\frac{72}{7}$, $r_2=\frac{63}{8}$, and $r_3=14$).

30. Denote the entries of M by m_{ij} , where $1 \leq i, j \leq 3$. The matrix has an inverse if its determinant is nonzero. We count the number of matrices M with a zero determinant, then subtract this from the total number of such matrices, which is $2^9=512$. To that end, let $D=\det(M)=m_{11}(m_{22}m_{33}-m_{23}m_{32})+m_{12}(m_{23}m_{31}-m_{21}m_{33})+m_{13}(m_{21}m_{32}-m_{22}m_{31})$. To make the argument a bit easier to follow, let $c_1=m_{22}m_{33}-m_{23}m_{32}$, $c_2=m_{23}m_{31}-m_{21}m_{33}$, and $c_3=m_{21}m_{32}-m_{22}m_{31}$, which makes $\det(M)=m_{11}c_1+m_{12}c_2+m_{13}c_3$. Now if $c_1=c_2=c_3=0$, then m_{11} , m_{12} , and m_{13} could be any element from $\{0,1\}$; thus, there are $2^3=8$ such ways that $D=0$ in this case. If $c_1 \neq 0$, $c_2 \neq 0$, and $c_3 \neq 0$ simultaneously, then either $m_{11}=m_{12}=m_{13}=0$ or the sum of two of the $m_{1k}c_k$'s is the opposite of the third; thus, there are 4 ways in this case.

Now we count the number of ways m_{21} , m_{22} , m_{23} , m_{31} , m_{32} , and m_{33} could give $c_1=c_2=c_3=0$. If $m_{21}=m_{22}=m_{23}=0$, then the other three entries could be any of 8 combinations, and we have $c_1=c_2=c_3=0$. There are 7 ways we could have $m_{21} \neq 0$, $m_{22} \neq 0$, and $m_{23} \neq 0$ simultaneously, and for each of these, there are 2 ways to assign values to m_{31} , m_{32} , and m_{33} so that $c_1=c_2=c_3=0$. (For example, suppose $m_{21}=m_{22}=m_{23}=1$. Then either $m_{31}=m_{32}=m_{33}=0$ or $m_{31}=m_{32}=m_{33}=1$. Likewise, $m_{21}=0$ and $m_{22}=m_{23}=1$, then either $m_{31}=m_{32}=m_{33}=0$ or $m_{31}=0$ and $m_{32}=m_{33}=1$. The other combinations work similarly.)

Thus, there are $8+7 \cdot 2=22$ ways m_{21} , m_{22} , m_{23} , m_{31} , m_{32} , and m_{33} could give $c_1=c_2=c_3=0$. Since there are $2^6=64$ possible ways to assign either 0 or 1 to the six entries, and 22 of them give $c_1=c_2=c_3=0$, then 42 of them do not. It follows that the total number of ways that

$D=0$ is $22 \cdot 8+4 \cdot 42=344$. Finally, the probability that $D \neq 0$ is $\frac{512-344}{512}=\frac{168}{512}=\frac{21}{64}$.