

Answers:

1. D
2. A
3. A
4. E
5. B
6. C
7. B
8. D
9. B
10. E
11. B
12. C
13. C
14. B
15. D
16. A
17. D
18. B
19. A
20. E
21. A
22. C
23. D
24. C
25. D
26. A
27. B
28. C
29. B
30. B

Solutions:

$$1. \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+4x} - \sqrt{1-2x}}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{(\sqrt{1+4x} - \sqrt{1-2x})(\sqrt{1+4x} + \sqrt{1-2x})}{x(\sqrt{1+4x} + \sqrt{1-2x})} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{(1+4x) - (1-2x)}{x(\sqrt{1+4x} + \sqrt{1-2x})} \right) = \lim_{x \rightarrow 0} \left(\frac{6x}{x(\sqrt{1+4x} + \sqrt{1-2x})} \right) = \frac{6}{\sqrt{1} + \sqrt{1}} = 3$$

$$2. \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\sin x + \cos x} = \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = \sin^2 x - \sin x \cos x + \cos^2 x$$

$$1 - \sin x \cos x = 1 - \frac{1}{2} \sin 2x \Rightarrow \frac{d}{dx} \left(1 - \frac{1}{2} \sin 2x \right) = -\cos 2x$$

$$3. \text{ Let } f(x) = \frac{e^{2x}}{\sqrt{x}}. \text{ Then } f'(x) = \frac{\sqrt{x} \cdot 2e^{2x} - e^{2x} \cdot \frac{1}{2\sqrt{x}}}{x}, \text{ and the unique critical point is } x = \frac{1}{4},$$

where the function attains a local minimum (the numerator of f grows much more quickly than the denominator). Therefore, plugging this value into f yields $f\left(\frac{1}{4}\right) = 2\sqrt{e}$.

4. Let $P(x) = ax^2 + bx + c$ and $Q(x) = dx^2 + ex + f$, where $a, d \neq 0$. If $Q(x_0) = 0$, then

$$P(0) = P(Q(x_0)) = P(x_0) \cdot Q(x_0) = P(x_0) \cdot 0 = 0 \Rightarrow c = 0 \Rightarrow P(x) = ax^2 + bx. \text{ Therefore,}$$

$$a(Q(x))^2 + b(Q(x)) = (ax^2 + bx) \cdot Q(x), \text{ and since } Q \text{ is not the zero polynomial,}$$

$$a(Q(x)) + b = ax^2 + bx \Rightarrow Q(x) = x^2 + \frac{b}{a}x - \frac{b}{a}, \text{ and substituting } k = \frac{b}{a}, Q(x) = x^2 + kx - k.$$

$$\text{Therefore, } 1 = 2025^2 + k \cdot 2025 - k \Rightarrow k = \frac{1 - 2025^2}{2025 - 1} = -2026, \text{ and } Q(2026) = 2026^2 - 2026 \cdot 2026 + 2026 = 2026.$$

$$5. V = \frac{4}{3} \pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 100 = 4\pi (25)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{25\pi}$$

$$6. \text{ Since } f'(x) = x^2 \cdot f(x), f''(x) = x^2 \cdot f'(x) + 2x \cdot f(x). \text{ Further, } f'(2) = 2^2 \cdot f(2) = 40, \text{ so}$$

$$f''(2) = 2^2 \cdot f'(2) + 2(2) \cdot f(2) = 200.$$

7. Parametrize the ellipse as $x = 3\cos\theta$ and $y = 2\sin\theta$ for $0 \leq \theta \leq 2\pi$ to get $f(x, y) = x^2 + xy$

$+ y^2 = 9\cos^2\theta + 6\sin\theta\cos\theta + 4\sin^2\theta = \frac{13}{2} + 3\sin 2\theta + \frac{5}{2}\cos 2\theta$. The oscillatory part,

$3\sin 2\theta + \frac{5}{2}\cos 2\theta$, has amplitude $\sqrt{3^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{61}}{2}$, so the maximum value is $M = \frac{13}{2} + \frac{\sqrt{61}}{2}$,

the minimum value is $m = \frac{13}{2} - \frac{\sqrt{61}}{2}$, and $M - m = \sqrt{61}$.

8. The y -coordinate of the tangency point is $f(2) = \sqrt{4(2)+1} = 3$. Further, $f(x) = (4x+1)^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2}(4x+1)^{-\frac{1}{2}}(4) = \frac{2}{\sqrt{4x+1}}$, so the slope of the tangent at this point is $f'(2)$

$= \frac{2}{\sqrt{4(2)+1}} = \frac{2}{3}$, and the tangent is $y - 3 = \frac{2}{3}(x - 2) \Rightarrow y = \frac{2}{3}x + \frac{5}{3}$, so the y -intercept is $\frac{5}{3}$.

9. For $x > 0$, let $f(x) = x^{\frac{1}{x}} \Rightarrow \ln f(x) = \frac{\ln x}{x} \Rightarrow \frac{f'(x)}{f(x)} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} \Rightarrow f'(x) = f(x) \cdot \frac{1 - \ln x}{x^2}$. The

only critical number for f is e , and $f'(x) > 0$ for $x < e$ while $f'(x) < 0$ for $x > e$, making the maximum value of f at $x = e$. Testing the two integers closest to that, $f(2) = \sqrt{2} = \sqrt[6]{8}$ and $f(3) = \sqrt[3]{3} = \sqrt[6]{9}$, so $m = 3$ gives the largest value of the expression among integers.

10. According to the recursion, $2025 = a_{111} = a_{55} = a_{27} = a_{13} = a_6 = a_5 + a_7 = a_2 + a_3 = (a_1 + a_3) + a_1 = (a_1 + a_1) + a_1 = 3a_1 \Rightarrow a_1 = 675$.

11. When $n \geq 2$, $I_n = \int_0^{\frac{\pi}{4}} (\tan^n x) dx = \int_0^{\frac{\pi}{4}} ((\tan^{n-2} x)(\tan^2 x)) dx = \int_0^{\frac{\pi}{4}} ((\tan^{n-2} x)(\sec^2 x - 1)) dx$

$$\int_0^{\frac{\pi}{4}} ((\tan^{n-2} x)(\sec^2 x)) dx - I_{n-2} = \frac{1}{n-1} \tan^{n-1} x \Big|_0^{\frac{\pi}{4}} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \Rightarrow I_n + I_{n-2} = \frac{1}{n-1}.$$

12. $\lim_{x \rightarrow 0^+} f(x) = 0 = \lim_{x \rightarrow 0^-} f(x)$ and $f(0) = 0$, so I is true. Also, $f'(x) = \begin{cases} 2x, & \text{if } x > 0 \\ -2x, & \text{if } x < 0 \end{cases}$ with

$\lim_{x \rightarrow 0^+} f'(x) = 0 = \lim_{x \rightarrow 0^-} f'(x) \Rightarrow f'(0) = 0$, so II is true, and $f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases}$ —this, however,

also shows that III is true. Lastly, $f'(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$ will not have a limit as $x \rightarrow 0$, so IV is false.

$$13. \text{ Letting } u = e^{-x}, du = -e^{-x}dx = -u dx \Rightarrow -\frac{du}{u} = dx, \text{ and substituting the limits, } \int_0^{\ln 2} \frac{1}{1+e^x} dx \\ = -\int_1^{\frac{1}{2}} \frac{\frac{1}{u}}{1+\frac{1}{u}} du = \int_{\frac{1}{2}}^1 \frac{1}{u+1} du = \ln|u+1| \Big|_{\frac{1}{2}}^1 = \ln 2 - \ln \frac{3}{2} = \ln \frac{4}{3}.$$

$$14. f(x) = \int_1^x te^{2t-x} dt = e^{-x} \int_1^x te^{2t} dt \Rightarrow f'(x) = e^{-x}(xe^{2x}) - e^{-x} \int_1^x te^{2t} dt \Rightarrow f'(1) = e.$$

$$15. \text{ The Maclaurin series for } f(x) = \cos(x^3) \text{ is } 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \dots, \text{ and these are} \\ \text{actually equal; i.e., } f(x) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots. \text{ By the sixth derivative, all terms from the 3}^{\text{rd}} \\ \text{term on have at least an } x \text{ in the expression, so they will evaluate to 0. Therefore, } f^{(6)}(0) \\ = -\frac{6!}{2} = -360.$$

$$16. \sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \ln\left(\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)\right) = \lim_{i \rightarrow \infty} \left(\ln\left(\frac{i+1}{2i}\right)\right) = \ln \frac{1}{2} = -\ln 2.$$

$$17. \text{ Use the half-angle identities } \sqrt{1+\sin x} = \left|\sin \frac{x}{2} + \cos \frac{x}{2}\right| \text{ and } \sqrt{1-\sin x} = \left|\sin \frac{x}{2} - \cos \frac{x}{2}\right|. \text{ Since} \\ \text{for } 0 \leq x \leq \pi, 0 \leq \frac{x}{2} \leq \frac{\pi}{2}, \text{ so across this entire interval, } \sqrt{1+\sin x} = \sin \frac{x}{2} + \cos \frac{x}{2}. \text{ On } 0 \leq x \leq \frac{\pi}{2}, \\ \sqrt{1-\sin x} = \cos \frac{x}{2} - \sin \frac{x}{2}, \text{ and on } \frac{\pi}{2} \leq x \leq \pi, \sqrt{1-\sin x} = \sin \frac{x}{2} - \cos \frac{x}{2}. \text{ Therefore,} \\ \int_0^{\pi} (\sqrt{1+\sin x} + \sqrt{1-\sin x}) dx = \int_0^{\frac{\pi}{2}} (\sqrt{1+\sin x} + \sqrt{1-\sin x}) dx + \int_{\frac{\pi}{2}}^{\pi} (\sqrt{1+\sin x} + \sqrt{1-\sin x}) dx \\ \int_0^{\frac{\pi}{2}} \left(2 \cos \frac{x}{2}\right) dx + \int_{\frac{\pi}{2}}^{\pi} \left(2 \sin \frac{x}{2}\right) dx = 4 \sin \frac{x}{2} \Big|_0^{\frac{\pi}{2}} + \left(-4 \cos \frac{x}{2}\right) \Big|_{\frac{\pi}{2}}^{\pi} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}.$$

$$18. \int_0^{\frac{\pi}{2}} (|\sin x - \cos x|) dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx = (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}} \\ + (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2.$$

$$19. \text{ Letting } x = \tan \theta, \, dx = \sec^2 \theta d\theta, \text{ and substituting the limits, } \int_0^{\infty} \frac{x^2}{(1+x^2)^2} dx \\ = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^2 \theta}{\sec^4 \theta} \cdot \sec^2 \theta \right) d\theta = \int_0^{\frac{\pi}{2}} (\sin^2 \theta) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta = \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

$$20. \text{ Since } P(k) = \frac{(k+1)^2}{k} \text{ for } k \text{ from } 1 \text{ to } 10, \, kP(k) - (k+1)^2 = 0 \text{ for these values, so} \\ xP(x) - (x+1)^2 \text{ is a degree } 10 \text{ polynomial with the integers from } 1 \text{ to } 10 \text{ as roots; i.e.,} \\ xP(x) - (x+1)^2 = a(x-1)(x-2)\dots(x-10) \text{ for some nonzero constant } a. \text{ Plugging in } x=0 \\ \text{ gives } -1 = 10!a \Rightarrow a = -\frac{1}{10!} \Rightarrow xP(x) - (x+1)^2 = -\frac{1}{10!}(x-1)(x-2)\dots(x-10). \text{ Now, plugging in} \\ x=12 \text{ gives } 12P(12) - 169 = -\frac{11!}{10!} = -11 \Rightarrow 12P(12) = 158.$$

$$21. \text{ The critical points of a cubic polynomial } f \text{ must satisfy } f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0, \text{ so to} \\ \text{have two distinct local extrema, this quadratic equation must have two distinct real zeros.} \\ \text{Therefore, the discriminant must be positive; i.e., } (2b)^2 - 4(3a)(c) > 0 \Rightarrow 4b^2 - 12ac > 0 \\ 4b^2 > 12ac \Rightarrow b^2 > 3ac.$$

$$22. \text{ Multiply both sides of the equation } g(x) = \sum_{n=1}^{\infty} F_n x^n \text{ by } x \text{ to get } xg(x) = \sum_{n=1}^{\infty} F_n x^{n+1}, \text{ then} \\ \text{subtract that amount from the original: } g(x) - xg(x) = \sum_{n=1}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_n x^{n+1} = F_1 x + \sum_{n=2}^{\infty} F_n x^n \\ - \sum_{n=1}^{\infty} F_n x^{n+1} = F_1 x + \sum_{n=1}^{\infty} F_{n+1} x^{n+1} - \sum_{n=1}^{\infty} F_n x^{n+1} = F_1 x + \sum_{n=1}^{\infty} (F_{n+1} - F_n) x^{n+1} = F_1 x + \sum_{n=1}^{\infty} F_{n-1} x^{n+1} \\ = F_1 x + x^2 \sum_{n=1}^{\infty} F_{n-1} x^{n-1} = F_1 x + x^2 \sum_{n=1}^{\infty} F_n x^n = x + x^2 g(x) \Rightarrow x = g(x) - xg(x) - x^2 g(x) = (1 - x - x^2)g(x) \\ \frac{x}{1 - x - x^2} = g(x).$$

$$23. \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}} x^{n+1}}{\frac{n!}{n^n} x^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n^n}{(n+1)^n} |x| \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{\left(1 + \frac{1}{n}\right)^n} |x| \right) = \frac{|x|}{e}, \text{ and this}$$

limit must be less than 1 for convergence, so $|x| < e$, making the radius of convergence e .

$$24. \text{ Let } S = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \sin \frac{n\pi}{3} \right). \text{ For } |a| < 1 \text{ and real } \theta, \sum_{n=1}^{\infty} (a^n e^{in\theta}) = \frac{ae^{i\theta}}{1-ae^{i\theta}}, \text{ the imaginary part of}$$

$$\text{which is } \sum_{n=1}^{\infty} (a^n \sin(n\theta)) = \frac{a \sin \theta}{1-2a \cos \theta + a^2}. \text{ Plugging in } a = \frac{1}{2} \text{ and } \theta = \frac{\pi}{3} \text{ gives } \frac{\frac{\sqrt{3}}{4}}{\frac{3}{4}} = \frac{\sqrt{3}}{3}.$$

$$25. \text{ Plugging in } x = 3 \text{ gives } f(3) + 2f\left(\frac{1}{3}\right) = 9, \text{ while plugging in } x = \frac{1}{3} \text{ gives } f\left(\frac{1}{3}\right) + 2f(3) = 1.$$

Adding the two equations together, then dividing by 3 gives $f\left(\frac{1}{3}\right) + f(3) = \frac{10}{3}$; subtracting this

equation from the second one gives $f(3) = -\frac{7}{3}$.

26. $f^3(n) = n+3 \Rightarrow f(n+3) = f^4(n) = f^3(f(n)) = f(n)+3$. Now, define $g(n) = f(n) - n$, meaning $g(n+3) = f(n+3) - (n+3) = f(n)+3 - (n+3) = f(n) - n = g(n)$, so g is 3-periodic. Writing $g(3k) = \alpha$, $g(3k+1) = \beta$, and $g(3k+2) = \gamma$, because the three numbers n , $f(n) = n+g(n)$, and $f^2(n) = n+g(n)+g(n+g(n))$ have distinct residues mod 3, summing for these residues and using $f^3(n) = n+3$ gives $\alpha + \beta + \gamma = 3$. Since f is a function on natural numbers, $3 + \alpha > 0 \Rightarrow \alpha \geq -2$; $1 + \beta > 0 \Rightarrow \beta \geq 0$; and $2 + \gamma > 0 \Rightarrow \gamma \geq -1$; further, injectivity of f forces these three images to be distinct, so there are exactly six possibilities for (α, β, γ) : $(-2, 1, 4)$, $(-2, 4, 1)$, $(-1, 2, 2)$, $(-1, 5, -1)$, $(1, 1, 1)$, and $(2, 2, -1)$. Since $100 \equiv 1 \pmod{3}$, $f(100) = 100 + \beta$, making the only possibilities for $f(100)$ 101, 102, 104, or 105. The sum of these options is 412.

27. Let $P(x) = x^6 + x^4 + x^2 + 1$. When P is divided by $(x-1)^3$, the remainder R is the quadratic Taylor polynomial of P about $x=1$: $R(x) = P(1) + P'(1)(x-1) + \frac{P''(1)}{2}(x-1)^2$. Now, since

$P'(x) = 6x^5 + 4x^3 + 2x$, $P'(1) = 12$; further, $P''(x) = 30x^4 + 12x^2 + 2$, making $P''(1) = 44$.

Therefore, $R(0) = P(1) - P'(1) + \frac{P''(1)}{2} = 4 - 12 + 22 = 14$.

28. A generic point on this parabola is labeled $\left(\frac{y^2}{2}, y\right)$, so the distance from this point to $(1, 4)$

is $d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y - 4)^2}$, and we need to minimize this expression. However, since this

function can't be negative, we can just minimize the radicand $f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y - 4)^2$.

$f'(y) = 2\left(\frac{y^2}{2} - 1\right)y + 2(y - 4) = y^3 - 8$, so the only critical point is $y = 2$. Since points on this parabola can continue to get farther and farther away from $(1, 4)$, this point gives a minimum

distance, which is $d = \sqrt{\left(\frac{2^2}{2} - 1\right)^2 + (2 - 4)^2} = \sqrt{5}$.

29. For $x^3 + y^3 - 3xy = 0$, $3x^2 + 3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$. For the

tangent to be horizontal then, we must have $y = x^2$, so $0 = x^3 + (x^2)^3 - 3x(x^2) = x^6 - 2x^3 = x^3(x^3 - 2)$, so the only point that fits the description has $a = \sqrt[3]{2} \Rightarrow b = \sqrt[3]{4} \Rightarrow ab = 2$.

30. Using the definition, $(f_3 \circ f_2 \circ f_1 \circ f_0)(x) = |||x| - 1| - 2| - 3|$, which is an even function that can

be written as a piecewise function
$$\begin{cases} 2 - x, & \text{if } 0 \leq x < 1 \\ x, & \text{if } 1 \leq x < 3 \\ 6 - x, & \text{if } 3 \leq x \leq 6 \end{cases}$$
 (evenness defines this function similarly

for negative x). Therefore, $\int_{-6}^6 ((f_3 \circ f_2 \circ f_1 \circ f_0)(x)) dx = 2 \int_0^6 ((f_3 \circ f_2 \circ f_1 \circ f_0)(x)) dx$

$$2 \left(\int_0^1 (2 - x) dx + \int_1^3 (x) dx + \int_3^6 (6 - x) dx \right) = 2 \left(\left(2x - \frac{x^2}{2} \right) \Big|_0^1 + \left(\frac{x^2}{2} \right) \Big|_1^3 + \left(6x - \frac{x^2}{2} \right) \Big|_3^6 \right) = 2 \left(\frac{3}{2} + 4 + \frac{9}{2} \right) = 20.$$