

Answers:

<u>Relay 1</u> 1. 4 2. -9 3. -5 4. 1 5. 27 6. 2	<u>Relay 4</u> 1. 39 2. 8 3. 169 4. 676 5. 28 6. 2
<u>Relay 2</u> 1. -33 2. 3 3. 71 4. -2 5. 6 6. -171	<u>Relay 5</u> 1. -6 2. -1 3. -40 4. -35 5. 210 6. 576
<u>Relay 3</u> 1. 9 2. 68 3. 99 4. 3796 5. 100 6. 5	

Solutions:

Relay 1

1. Due to the continuity of polynomials, $\lim_{x \rightarrow 3} (2x^3 - 7x^2 + 4x + 1) = 2(3)^3 - 7(3)^2 + 4(3) + 1$
 $= 54 - 63 + 12 + 1 = 4$.

2. Due to the continuity of rational functions at numbers in their domains, $\lim_{x \rightarrow 2} \left(\frac{x^2 + 4x - 3}{x^2 - 4x + 3} \right)$
 $= \frac{(2)^2 + 4(2) - 3}{(2)^2 - 4(2) + 3} = \frac{4 + 8 - 3}{4 - 8 + 3} = \frac{9}{-1} = -9$.

3. Due to the continuity of rational functions at numbers in their domains, $\lim_{x \rightarrow 3} \left(\frac{-9x^2 + 24x + 9}{x^2 - 9} \right)$
 $= \lim_{x \rightarrow 3} \left(\frac{(-9x - 3)(\cancel{x - 3})}{(x + 3)(\cancel{x - 3})} \right) = \lim_{x \rightarrow 3} \left(\frac{-9x - 3}{x + 3} \right) = \frac{-9(3) - 3}{(3) + 3} = \frac{-30}{6} = -5$.

4. $\lim_{h \rightarrow 0} \left(\frac{(2+h)^2 - 3(2+h) + 7 - 5}{h} \right)$ represents the derivative of the function $f(x) = x^2 - 3x + 7$ at
 $x = 2$. Since this derivative is $f'(x) = 2x - 3$, $f'(2) = 2(2) - 3 = 4 - 3 = 1$.

5. Because this rational expression is defined as $x \rightarrow \infty$ and has the same degree in both the numerator and the denominator, the limit is the ratio of leading coefficients. Therefore,

$$\lim_{x \rightarrow \infty} \left(\frac{27x^3 - 2x^2 + 10x - 5}{x^3 - 2x^2 + 15x - 6} \right) = \frac{27}{1} = 27.$$

6. Both the numerator and denominator are defined for $x \rightarrow -\infty$, but for $x \rightarrow -\infty$,

$$\begin{aligned} \sqrt{81x^2 - 7x + 7} &\approx 9|x| = -9x. \text{ Therefore, } \lim_{x \rightarrow -\infty} \left(\frac{-27x + \sqrt{81x^2 - 7x + 7}}{-27x - \sqrt{81x^2 - 7x + 7}} \right) = \lim_{x \rightarrow -\infty} \left(\frac{-27x - 9x}{-27x + 9x} \right) \\ &= \lim_{x \rightarrow -\infty} \left(\frac{-36x}{-18x} \right) = \frac{-36}{-18} = 2. \end{aligned}$$

Relay 2

1. For this function, $\frac{dy}{dx} = -6x^2 + 8x - 3 \Rightarrow \frac{dy}{dx}\bigg|_{x=-\frac{5}{3}} = -6\left(-\frac{5}{3}\right)^2 + 8\left(-\frac{5}{3}\right) - 3 = -\frac{50}{3} - \frac{40}{3} - \frac{9}{3} = -\frac{99}{3}$

$= -33$.

2. Using the information found in the previous problem, $-6x^2 + 8x - 3 = -33 \Rightarrow 0 = -6x^2 + 8x + 30 = -2(3x^2 - 4x - 15) = -2(3x + 5)(x - 3)$, so the other x -value is 3.

3. The y -value of the point on this function where $x = 3$ is $-2(3)^3 + 4(3)^2 - 3(3) - 1 = -54 + 36 - 9 - 1 = -28$, so using the information found in the previous parts of this problem, the equation of the sought tangent line is $y + 28 = -33(x - 3) \Rightarrow y = -33x + 71$. Therefore, the y -intercept of this line is 71.

4. Since $\frac{dy}{dx} = -6x^2 + 8x - 3$ has no real zeros (the discriminant of this quadratic expression is $8^2 - 4(-6)(-3) = 64 - 72 = -8 < 0$), along with the end behavior of a cubic polynomial with negative leading coefficient, this function is always decreasing, meaning there is only one such point. Since $y(-3) = -2(-3)^3 + 4(-3)^2 - 3(-3) - 1 = 54 + 36 + 9 - 1 = 98 > 71$ and $y(-2) = -2(-2)^3 + 4(-2)^2 - 3(-2) - 1 = 16 + 16 + 6 - 1 = 37 < 71$, since this function is a polynomial and therefore continuous, by the Intermediate Value Theorem, there is an x -value c in the interval $[-3, -2]$ such that $y(c) = 71$. Therefore, $A = -2$.

5. By the Mean Value Theorem, $f'(r) = \frac{f(Z) - f(0)}{Z - 0} = \frac{(-2Z^3 + 4Z^2 - 3Z - 1) - (-1)}{Z}$
 $= \frac{-2Z^3 + 4Z^2 - 3Z}{Z} = -2Z^2 + 4Z - 3$, so $-2Z^2 + 4Z - 3 = -2 \Rightarrow 0 = 2Z^2 - 4Z + 1$
 $\Rightarrow Z = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2}$, and both of these are greater than 0, so the greater possible answer is $\frac{2 + \sqrt{2}}{2}$. This answer is in the desired form, so $B + C + D = 2 + 2 + 2 = 6$.

6. For this function, $\frac{dy}{dx} = -6x^2 + 8x - 3 \Rightarrow \frac{dy}{dx}\bigg|_{x=6} = -6(6)^2 + 8(6) - 3 = -216 + 48 - 3 = -171$.

Relay 3

1. This function is continuous on the interval, and $f'(x) = 2 - \frac{4}{x^2}$, so the critical number in the given interval is $x = \sqrt{2}$. According to the Extreme Value Theorem, the maximum value of the function on this interval occurs either at this critical number or at one of the endpoints. $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) + \frac{4}{\left(\frac{1}{2}\right)} = 1 + 8 = 9$, $f(2) = 2(2) + \frac{4}{(2)} = 4 + 2 = 6$, and $f(\sqrt{2}) = 2(\sqrt{2}) + \frac{4}{\sqrt{2}} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} \approx 5.6...$, so the maximum value of this function on this interval is 9.

2. Let x and y be the distances that Cars A and B are from the intersection, respectively, and let z be the distance between the cars. Since the angle is a right angle, we have $x^2 + y^2 = z^2$. $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = z \frac{dz}{dt}$. Further, when the cars are 12 and 9 miles from the intersection, the distance between the cars is 15 miles. Therefore, $(12)(-40) + (9)(-60) = (15) \frac{dz}{dt} \Rightarrow \frac{dz}{dt} = -68$. Therefore, the distance between the cars is decreasing a rate of 68 miles per hour.

3. Since the two borders of the dartboard intersect at $x=1$ and $x=5$, the area of the dartboard is $\int_1^5 (-x^2 + 6x - 5) dx = \left(-\frac{1}{3}x^3 + 3x^2 - 5x \right) \Big|_1^5 = \left(-\frac{1}{3}(5)^3 + 3(5)^2 - 5(5) \right) - \left(-\frac{1}{3}(1)^3 + 3(1)^2 - 5(1) \right) = -\frac{125}{3} + 75 - 25 + \frac{1}{3} - 3 + 5 = \frac{32}{3}$. The line $y=3$ intersects the parabola at $x=2$ and $x=4$, so the area above $y=3$ is $\int_2^4 ((-x^2 + 6x - 5) - 3) dx = \int_2^4 (-x^2 + 6x - 8) dx = \left(-\frac{1}{3}x^3 + 3x^2 - 8x \right) \Big|_2^4 = \left(-\frac{1}{3}(4)^3 + 3(4)^2 - 8(4) \right) - \left(-\frac{1}{3}(2)^3 + 3(2)^2 - 8(2) \right) = -\frac{64}{3} + 48 - 32 + \frac{8}{3} - 12 + 16 = \frac{4}{3}$, making the probability of the dart landing above the line $\frac{\frac{4}{3}}{\frac{32}{3}} = \frac{4}{32} = \frac{1}{8}$. Therefore, the probability of the dart landing below the line is $1 - \frac{1}{8} = \frac{7}{8}$, and this fraction is in lowest terms, so $5c + 8d = 5 \cdot 7 + 8 \cdot 8 = 35 + 64 = 99$.

$$4. \int_{-47}^{99} (x) dx = \frac{1}{2} x^2 \Big|_{-47}^{99} = \frac{1}{2} (99)^2 - \frac{1}{2} (-47)^2 = \frac{9801}{2} - \frac{2209}{2} = \frac{7592}{2} = 3796$$

$$5. S = 3 + 7 + 9 + 6 = 25, \text{ so } \int_{-1}^3 (25) dx = 25(3 - (-1)) = 25 \cdot 4 = 100$$

6. Such integers exist as long as k is a multiple of the greatest common divisor of $T = 100$ and 35 , so the least possible positive value of k is the greatest common divisor of 100 and 35 , which is 5 . (It is clear to see that because 100 and 35 are both multiples of 5 , k must also be a multiple of 5 . Further $m = -1$ and $n = 3$ will give this value, though it is unnecessary to find these values for this problem.)

Relay 4

$$1. \int_2^5 (x^2) dx = \left(\frac{1}{3} x^3 \right) \Big|_2^5 = \frac{1}{3} (5)^3 - \frac{1}{3} (2)^3 = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$$

$$2. \text{ First, } \tan \frac{\pi}{12} = \tan \frac{\frac{\pi}{6}}{2} = \frac{1 - \cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}. \text{ Using this information, } \int_{2-\sqrt{3}}^{\infty} \left(\frac{1}{1+x^2} \right) dx$$

$$= \lim_{t \rightarrow \infty} \int_{2-\sqrt{3}}^t \left(\frac{1}{1+x^2} \right) dx = \lim_{t \rightarrow \infty} \left(\tan^{-1} x \right) \Big|_{2-\sqrt{3}}^t = \lim_{t \rightarrow \infty} \left(\tan^{-1} t - \tan^{-1} (2 - \sqrt{3}) \right) = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}, \text{ so } A = 5$$

and $B = 12$. Therefore, when $39 + 5 = 44$ is divided by 12 , the quotient is 3 and the remainder is 8 .

$$3. \int_2^8 \left(\frac{4x}{1+x^2} \right) dx = \left(2 \ln(1+x^2) \right) \Big|_2^8 = 2 \ln(1+(8)^2) - 2 \ln(1+(2)^2) = 2 \ln 65 - 2 \ln 5 = 2 \ln \frac{65}{5} = 2 \ln 13$$

$$= \ln 13^2 = \ln 169, \text{ so } C = 169.$$

$$4. \text{ Interpreting in terms of area, } \int_0^{13} \left(\sqrt{169 - x^2} \right) dx \text{ is the area enclosed by the circle with}$$

equation $x^2 + y^2 = 169$ that is in the first quadrant, so the area is $\frac{1}{4} (\pi \cdot 13^2) = \frac{169\pi}{4}$. Therefore,

$D = 169$ and $E = 4$, meaning $D \cdot E = 169 \cdot 4 = 676$.

5. The two graphs intersect when $x = 0$ and $x = 6$, with the first function forming the upper boundary of the region, so the enclosed area of the region is $\int_0^6 ((12x - x^2) - (x^2)) dx$

$$= \int_0^6 (12x - 2x^2) dx = \left(6x^2 - \frac{2}{3}x^3 \right) \Big|_0^6 = \left(6(6)^2 - \frac{2}{3}(6)^3 \right) - \left(6(0)^2 - \frac{2}{3}(0)^3 \right) = 216 - 144 - 0 + 0 = 72.$$

Therefore, when 676 is divided by 72, the quotient is 9 and the remainder is 28.

$$6. 28 = \int_G^H (x) dx = \left(\frac{1}{2}x^2 \right) \Big|_G^H = \frac{1}{2}(H^2 - G^2) \Rightarrow 56 = (H - G)(H + G), \text{ so if } H \text{ and } G \text{ are positive}$$

integers, it must be the case that $H > G$ and $H - G < H + G$. The only ways to factor 56 using positive integers are $1 \cdot 56$, $2 \cdot 28$, $4 \cdot 14$, and $7 \cdot 8$. Setting $H - G$ equal to the smaller factor and $H + G$ equal to the larger factor, the first and last factorizations do not have integer solutions ($(27.5, 28.5)$ and $(0.5, 7.5)$, respectively). However, the second and third factorizations do have integer solutions ($(13, 15)$ and $(5, 9)$, respectively). Therefore, there are two such ordered pairs, listed above.

Relay 5

1. The solutions to this differential equation are all of the form $y = x^2 + C$ for some constant C . Using the point given, $-3 = 2^2 + C \Rightarrow C = -7$, so $f(x) = x^2 - 7$, making $f(1) = 1^2 - 7 = 1 - 7 = -6$.

2. The solutions to this differential equation are all of the form $y = -\frac{1}{x} + C$ for some constant C .

Using the point given, $-6 = -\frac{1}{-6} + C = \frac{1}{6} + C \Rightarrow C = -\frac{37}{6}$, so $g(x) = -\frac{1}{x} - \frac{37}{6}$, making $g\left(-\frac{6}{31}\right)$

$$= -\frac{1}{-\frac{6}{31}} - \frac{37}{6} = \frac{31}{6} - \frac{37}{6} = -\frac{6}{6} = -1.$$

3. The average value of h on the interval $[-1, 4]$ is $\frac{1}{4 - (-1)} \int_{-1}^4 h(x) dx$

$$= \frac{1}{5} \int_{-1}^4 (-4x^3 + 3x^2 - 2x + 1) dx = \frac{1}{5} \left(-x^4 + x^3 - x^2 + x \right) \Big|_{-1}^4 = \frac{1}{5} \left(\left(-(4)^4 + (4)^3 - (4)^2 + (4) \right) - \left(-(-1)^4 + (-1)^3 - (-1)^2 + (-1) \right) \right) = \frac{1}{5} (-256 + 64 - 16 + 4 + 1 + 1 + 1 + 1) = \frac{1}{5} (-200) = -40.$$

4. The infinite geometric series has first term -40 and common ratio $\frac{-40}{280} = -\frac{1}{7}$, so the sum is

$$\frac{-40}{1 - \left(-\frac{1}{7}\right)} = \frac{-40}{1 + \frac{1}{7}} = \frac{-40}{\frac{8}{7}} = -35.$$

5. Parametrizing the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (-35)^2 = 1225$ using $x^{\frac{2}{3}} = t$, $t + y^{\frac{2}{3}} = 1225$

$\Rightarrow y^{\frac{2}{3}} = 1225 - t$, and solving these equations for x and y , $x = t^{\frac{3}{2}}$ and $y = (1225 - t)^{\frac{3}{2}}$. Therefore,

$\frac{dx}{dt} = \frac{3}{2}t^{\frac{1}{2}}$ and $\frac{dy}{dt} = -\frac{3}{2}(1225 - t)^{\frac{1}{2}}$. Since the y -intercept and the point where $x = 8$ correspond

to $t = 0$ and $t = 4$, respectively, the length of the curve between these points is

$$\begin{aligned} \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_0^4 \sqrt{\frac{9}{4}t + \frac{9}{4}(1225 - t)} dt = \int_0^4 \sqrt{\frac{9}{4}(1225)} dt = \int_0^4 \frac{105}{2} dt = \frac{105}{2}(4 - 0) \\ &= \frac{105}{2} \cdot 4 = 210. \end{aligned}$$

6. Since the prime factorization of 210 is $210 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$, the sum of the positive integral divisors of 210 is $(1 + 2)(1 + 3)(1 + 5)(1 + 7) = 3 \cdot 4 \cdot 6 \cdot 8 = 576$.