

Where applicable, "E) NOTA" indicates that none of the above answers is correct.

1. An arithmetic sequence $\{a_n\}_{n \geq 0}$ satisfies $a_8 = 10$ and $a_{12} = 30$. Find a_3 .

- A) $-\frac{70}{3}$ B) -90 C) -15 D) 0 E) NOTA

2. Let $\{a_n\}_{n \geq 0}$ be a sequence such that $a_{n+1} - 2a_n + a_{n-1}$ is constant for all $n \geq 1$, $a_0 = 0$, $a_8 = 10$, and $a_{12} = 30$. Find a_3 .

- A) -1 B) $-\frac{15}{16}$ C) $\frac{15}{16}$ D) 1 E) NOTA

3. Clinton asks you for the set of all x such that $\sum_{n=0}^{\infty} \left(\frac{n^n}{n!} \left(\frac{1-x}{1+x} \right)^n \right)$ converges. What is the correct answer?

- A) $\{0\}$ B) $\left[-\frac{1}{e}, \frac{1}{e}\right)$ C) $\left(1 - \frac{e}{2}, 1 + \frac{e}{2}\right]$ D) $\left(\frac{e-1}{e+1}, \frac{e+1}{e-1}\right]$ E) NOTA

4. The exponential integral is defined as $\text{Ei}(x) = \int_1^x \left(\frac{e^{-t}}{t} \right) dt$. Find the second-degree Taylor polynomial for $\text{Ei}(x)$ centered at $a = 1$.

- A) $-\frac{1}{e}(x^2 - 3x + 2)$ B) $-\frac{1}{2e}(5x^2 - 14x + 11)$ C) $\frac{1}{2e}(x^2 - 4x + 5)$ D) $\frac{1}{e}(x^2 - 8x + 7)$
E) NOTA

5. Evaluate: $\sum_{k=0}^{\infty} \left(\frac{k^3}{k!} \right)$

- A) $3e$ B) $4e$ C) $5e$ D) $6e$ E) NOTA

For questions 6-9, use the following information: we say a series $\sum_{n=1}^{\infty} a_n$ *diverges absolutely*

when $\left| \sum_{n=1}^N a_n \right| \rightarrow \infty$ as $N \rightarrow \infty$, and we say the series *diverges conditionally* when the series does

not converge but $\left\{ \left| \sum_{n=1}^N a_n \right| \mid N \in \mathbb{N} \right\}$ is bounded.

6. Which of the following is an appropriate description regarding the convergence/divergence

of the series $\sum_{n=0}^{\infty} \left(\frac{(-2)^n}{2^{2^n}} \right)$?

- | | | |
|-------------------------|----------------------------|---------|
| A) converges absolutely | B) converges conditionally | |
| C) diverges absolutely | D) diverges conditionally | E) NOTA |

7. Which of the following is an appropriate description regarding the convergence/divergence

of the series $\sum_{n=1}^{\infty} \left((-1)^n \cdot \zeta(n+2) \right)$, where $\zeta(n) = \sum_{k=1}^{\infty} \left(\frac{1}{k^n} \right)$?

- | | | |
|-------------------------|----------------------------|---------|
| A) converges absolutely | B) converges conditionally | |
| C) diverges absolutely | D) diverges conditionally | E) NOTA |

8. Which of the following is an appropriate description regarding the convergence/divergence

of the series $\sum_{n=1}^{\infty} \left((-1)^n \cdot \frac{(\ln n)^{100}}{\sqrt{n}} \right)$?

- | | | |
|-------------------------|----------------------------|---------|
| A) converges absolutely | B) converges conditionally | |
| C) diverges absolutely | D) diverges conditionally | E) NOTA |

9. Which of the following is an appropriate description regarding the convergence/divergence

of the series $\sum_{n=1}^{\infty} \left(\frac{\text{NINELESS}(n)}{n} \right)$, where $\text{NINELESS}(n) = \begin{cases} 1, & \text{if } n \text{ contains no digit 9 in base 10} \\ 0, & \text{otherwise} \end{cases}$?

- | | | |
|-------------------------|----------------------------|---------|
| A) converges absolutely | B) converges conditionally | |
| C) diverges absolutely | D) diverges conditionally | E) NOTA |

10. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \begin{cases} e^{-\frac{1}{x^4}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Which of the following hold?

- I) f is smooth
- II) for every $x_0 \in \mathbb{R}$, the radius of convergence of the Taylor series for f at x_0 is positive
- III) for every $x_0 \in \mathbb{R}$, the Taylor series for f at x_0 converges to f in a neighborhood of x_0
- IV) some derivative of f is discontinuous

A) I and III only B) IV only C) I and II only D) I, II, and III only E) NOTA

11. Which of the following statements are true for sequences of real numbers $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$?

- I) If $\sum_{n=0}^{\infty} (|a_n|^2) < \infty$ and $\sum_{n=0}^{\infty} (|b_n|^2) < \infty$, then $\sum_{n=0}^{\infty} (a_n b_n)$ converges absolutely.
- II) If $\sum_{n=0}^{\infty} (a_n)$ converges, then $\lim_{n \rightarrow \infty} (n|a_n|) = 0$.
- III) If $a_n \geq a_{n+1}$ for every n , and if $\sum_{n=0}^N (b_n) \leq 1$ for every N , then $\sum_{n=0}^{\infty} (a_n b_n)$ converges.

A) II and III only B) I, II, and III C) I and II only D) I and III only E) NOTA

12. Suppose a power series $f(x) = \sum_{n=0}^{\infty} (a_n x^n)$ satisfies the differential equation $f'(x) = f(2x)$ with $f(0) = 1$. Determine the radius of convergence of the series that defines f .

A) 0 B) $\frac{1}{2}$ C) 2 D) ∞ E) NOTA

13. A sequence $\{a_n\}_{n \geq 0}$ satisfies the linear recurrence $\begin{cases} a_{n+2} = 3a_{n+1} - 2a_n \\ a_0 = 0 \\ a_1 = 1 \end{cases}$. Compute the value of $\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)$.

A) 1 B) 2 C) 3 D) 4 E) NOTA

For questions 20-22, the following information will be necessary: Let X be a random quantity which potentially takes values in a set $A = \{a_0, a_1, a_2, \dots\}$. The *expected value* of X is defined to be $E(X) = \sum_{\alpha \in A} (\alpha \cdot P(X = \alpha))$, where $P(X = \alpha)$ denotes the probability that X has taken on the value α . It is a theorem, which you may take for granted, that $E(\lambda X + X') = \lambda E(X) + E(X')$ for all random quantities X and X' and real constant λ .

20. You're offered to play an odd game: you flip a fair coin until you get tails, and win 1.5^j dollars, where j is the number of heads you flipped. Say you're willing to pay x dollars to enter (x is a whole number) **if and only if** your expected payout is greater than or equal to x . What is the most you should be willing to pay to enter this game?

- A) \$0 B) \$4 C) \$9 D) no maximum—you would play for any amount
E) NOTA

21. You're offered to play another odd game of a very similar nature: you flip a fair coin until you get tails, and you win 2^j dollars, where j is the number of heads you flipped. Letting X be your expected payout, your friend quickly computes that $E(X) = \infty$, and urges you to pay any amount the gamemaster asks in order to enter the game. You're not so keen on spending your life's savings on a parlor game, so you decide on another option: you're willing to pay x dollars to enter **if and only if** $E(\ln X) \geq x$. What's the most you're willing to pay to enter this game?

- A) \$2 B) \$2ln2 C) \$ln2 D) no maximum—you would play for any amount
E) NOTA

22. If Wikipedia is to be believed, there are forty-two notable restaurant chains in Poland. Tomasz makes it his missing to eat at every single one. He does so by picking, on any given day, one of the 42 chains uniformly at random at which to eat. Let N_k denote the number of **unique** restaurants visited after k days. Compute $E(N_k)$.

- A) $42 \left(1 - \left(1 - \frac{1}{42} \right)^k \right)$ B) $42 \left(1 - \frac{1}{42} \right)^k$ C) $42 \sum_{j=1}^n \left(\frac{1}{j} \right)$ D) $42 - \frac{(42-1)}{k}$ E) NOTA

23. Find the sum: $\sum_{j=1}^{\infty} \left(\frac{2}{2^j + 2^{j-1}} \right)$

- A) 1 B) $\frac{3}{2}$ C) $\frac{4}{3}$ D) $\frac{5}{4}$ E) NOTA

24. Find the sum: $\sum_{j=1}^{\infty} \left(\frac{2}{\sum_{i=0}^{\infty} (2^{j-i})} \right)$

- A) 1 B) $\frac{3}{2}$ C) $\frac{4}{3}$ D) $\frac{5}{4}$ E) NOTA

25. Find the sum: $\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(\frac{1}{2^{i+2j+3k}} \right) \right) \right)$

- A) $\frac{1}{48}$ B) $\frac{1}{32}$ C) $\frac{1}{24}$ D) $\frac{1}{21}$ E) NOTA

26. Find the sum: $\sum_{i=1}^{\infty} \left(\sum_{j=1}^i \left(\frac{3}{2^{i+2j}} \right) \right)$

- A) $\frac{3}{4}$ B) $\frac{4}{5}$ C) $\frac{5}{6}$ D) $\frac{6}{7}$ E) NOTA

27. Define a sequence $\{a_n\}_{n \geq 1}$ such that $a_1 = -\frac{13}{3}$, $a_2 = -\frac{19}{9}$, and $a_n = \frac{a_{n-1} + 2a_{n-2}}{3}$. Find the value of $\lim_{n \rightarrow \infty} (a_n)$.

- A) $-\frac{23}{8}$ B) -3 C) $-\frac{23}{7}$ D) $-\frac{35}{11}$ E) NOTA

28. Suppose that a and b are nonzero real numbers such that $\frac{\sum_{n=1}^{\infty} (a^n)}{\sum_{n=1}^{\infty} (b^n)} = \sum_{n=1}^{\infty} \left(\left(\frac{a}{b} \right)^n \right)$, where all

three individual series in this expression converge. For those real numbers a and b , all values in the intervals $(m, 0) \cup (0, n)$ are possible for the evaluation of either side of this equation. Find the least possible value of the product mn .

- A) $\frac{2-e}{e}$ B) $-\frac{17}{48}$ C) $-\frac{\sqrt{2}}{4}$ D) $\frac{1-\sqrt{5}}{3}$ E) NOTA

29. The sum of the first four terms of the infinite series $\sum_{n=1}^{\infty} \left((-1)^{n+1} \left(\frac{1}{n^2 + n} \right) \right)$ provides an estimate for the sum of the full series. Use this value and the alternating series test estimate to find an interval in which the sum of the infinite series lies. What is that interval?

- A) $\left[\frac{1}{3}, \frac{2}{5} \right]$ B) $\left[\frac{7}{20}, \frac{9}{20} \right]$ C) $\left[\frac{3}{10}, \frac{13}{30} \right]$ D) $\left[\frac{9}{25}, \frac{11}{25} \right]$ E) NOTA

30. The sum of the first four terms of the infinite series $\sum_{n=1}^{\infty} \left(\frac{1}{n^3 + n} \right)$ provides an estimate for the sum of the full series. Use this value and the integral test estimate to find an interval in which the sum of the infinite series lies. What is that interval?

- A) $\left[\frac{11}{17} + \ln \frac{\sqrt{37}}{5}, \frac{11}{17} + \ln \frac{\sqrt{26}}{4} \right]$ B) $\left[\frac{11}{17} + \ln \frac{\sqrt{26}}{5}, \frac{11}{17} + \ln \frac{\sqrt{17}}{4} \right]$
 C) $\left[\frac{661}{1020} + \ln \frac{\sqrt{37}}{5}, \frac{661}{1020} + \ln \frac{\sqrt{26}}{4} \right]$ D) $\left[\frac{661}{1020} + \ln \frac{\sqrt{26}}{5}, \frac{661}{1020} + \ln \frac{\sqrt{17}}{4} \right]$ E) NOTA