

Answers:

1. C
2. B
3. D
4. A
5. C
6. A
7. D
8. B
9. A
10. C
11. E
12. A
13. B
14. E
15. D
16. A
17. C
18. B
19. B
20. E
21. C
22. A
23. C
24. A
25. D
26. D
27. B
28. C
29. A
30. D

Solutions:

$$1. 30 = a_{12} = a_8 + (12-8)d = 10 + 4d \Rightarrow 20 = 4d \Rightarrow 5 = d \Rightarrow a_3 = a_8 + (3-8)d = 10 - 5(5) = 10 - 25 = -15$$

2. Since the second-level differences are constant, a_n is quadratic in terms of n , say $a_n = an^2 + bn + c$. Since $a_0 = 0$, $c = 0$, and since $a_8 = 10$ and $a_{12} = 30$, we have $10 = 64a + 8b$ and $30 = 144a + 12b$. Solving this system yields $a = \frac{5}{16}$ and $b = -\frac{5}{4}$, so $a_3 = \frac{5}{16}(3)^2 - \frac{5}{4}(3) = -\frac{15}{16}$.

$$3. \text{ Using the Ratio test, as } n \rightarrow \infty, \left| \frac{\frac{(n+1)^{n+1} \left(\frac{1-x}{1+x}\right)^{n+1}}{(n+1)!}}{\frac{n^n \left(\frac{1-x}{1+x}\right)^n}{n!}} \right| = \frac{(n+1)^n \left|\frac{1-x}{1+x}\right|}{n^n \left|\frac{1-x}{1+x}\right|} \rightarrow e \left|\frac{1-x}{1+x}\right|, \text{ so}$$

$$e \left|\frac{1-x}{1+x}\right| < 1 \Rightarrow \left|\frac{1-x}{1+x}\right| < \frac{1}{e} \Rightarrow -\frac{1}{e} < \frac{1-x}{1+x} < \frac{1}{e} \Rightarrow \frac{e-1}{e+1} < x < \frac{e+1}{e-1}. \text{ Now for the endpoints, use}$$

$$\text{Stirling's approximation: } \frac{n^n \left(\frac{1-x}{1+x}\right)^n}{n!} \approx \frac{n^n \left(\frac{1-x}{1+x}\right)^n}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = \frac{e^n \left(\frac{1-x}{1+x}\right)^n}{\sqrt{2\pi n}}. \text{ When } x = \frac{e-1}{e+1},$$

$$\frac{1-x}{1+x} = \frac{1}{e}, \text{ so } \frac{n^n \left(\frac{1-x}{1+x}\right)^n}{n!} \approx \frac{1}{\sqrt{2\pi n}}, \text{ which is a } p\text{-series that diverges. When } x = \frac{e+1}{e-1},$$

$$\frac{1-x}{1+x} = -\frac{1}{e}, \text{ so } \frac{n^n \left(\frac{1-x}{1+x}\right)^n}{n!} \approx \frac{(-1)^n}{\sqrt{2\pi n}}, \text{ which is a } p\text{-series that converges. Hence, the interval of}$$

$$\text{convergence is } \left[\frac{e-1}{e+1}, \frac{e+1}{e-1} \right].$$

$$4. \text{ Ei}(x) = \int_1^x \left(\frac{e^{-t}}{t} \right) dt \Rightarrow \text{Ei}'(x) = \frac{e^{-x}}{x} \Rightarrow \text{Ei}''(x) = \frac{-xe^{-x} - e^{-x}}{x^2}, \text{ so the second-degree Taylor}$$

$$\text{polynomial for this function around } a=1 \text{ is } \text{Ei}(1) + \text{Ei}'(1)(x-1) + \frac{\text{Ei}''(1)}{2}(x-1)^2 = 0 + e^{-1}(x-1)$$

$$-e^{-1}(x-1)^2 = -e^{-1}(x^2 - 3x + 2) = -\frac{1}{e}(x^2 - 3x + 2).$$

5. We use a classic trick. Starting from a general power series $f = \sum_{k=0}^{\infty} a_k x^k$, observe that

$$\left(x \frac{d}{dx}\right) f = \sum_{k=0}^{\infty} k a_k x^k \Rightarrow \left(x \frac{d}{dx}\right)^3 f = \sum_{k=0}^{\infty} k^3 a_k x^k. \text{ Applying this to } e^x = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right), \text{ we find that}$$

$$\sum_{k=0}^{\infty} \left(\frac{k^3}{k!}\right) x^k = \left(x \frac{d}{dx}\right)^3 e^x = \left(x \frac{d}{dx}\right)^2 x e^x = \left(x \frac{d}{dx}\right) (x^2 + x) e^x = x(x^2 + 3x + 1) e^x. \text{ Plugging in } x=1$$

$$\text{gives } \sum_{k=0}^{\infty} \left(\frac{k^3}{k!}\right) = 5e.$$

6. Eventually $\frac{2^n}{2^{2^n}} \leq \frac{100}{2^n}$, so by comparison with a geometric series, we find that this series converges absolutely.

7. The series certainly doesn't converge. Observe that $\lim_{n \rightarrow \infty} (\zeta(n+2)) = 1$, so $\{\zeta(n+2) - 1\}_n$ is a bounded sequence (in fact, it's bounded below by 0 and above by 2, but for now let's just say it's bounded above and below by M and $-M$, respectively, where $M > 0$). Therefore, we have

$$-2M \leq -M \sum_{n=1}^N (-1)^n \leq \sum_{n=1}^N ((-1)^n (\zeta(n+2) - 1)) \leq M \sum_{n=1}^N (-1)^n \leq 2M; \text{ hence,}$$

$$-3M \leq \sum_{n=1}^N ((-1)^n (\zeta(n+2))) \leq 3M \text{ for every } N, \text{ making this series diverge conditionally.}$$

8. The sequence $\left\{ \frac{(\ln n)^{100}}{\sqrt{n}} \right\}_{n \geq 1}$ is eventually decreasing to 0, so by the alternating series test,

this series converges. It does not converge absolutely, though, since for all but finitely many

values of n , $\frac{(\ln n)^{100}}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$, so by comparison to this p -series, the convergence is not absolute,

and thus the series is conditionally convergent.

9. Call a number n *nineless* when $\text{NINELESS}(n) = 1$. Group terms by the number of digits in the denominator; in general, there are $8 \cdot 9^{n-1}$ nineless n -digit numbers—equivalently, $8 \cdot 9^k$ nineless numbers between 10^k and $10^{k+1} - 1$. So for each $k \geq 0$, we have

$$\sum_{n=10^k}^{10^{k+1}-1} \left(\frac{\text{NINELESS}(n)}{n} \right) \leq \frac{8 \cdot 9^k}{10^k}. \text{ Therefore, } \sum_{n=1}^{\infty} \left(\frac{\text{NINELESS}(n)}{n} \right) = \sum_{k=0}^{\infty} \left(\sum_{n=10^k}^{10^{k+1}-1} \left(\frac{\text{NINELESS}(n)}{n} \right) \right)$$

$$\leq 8 \sum_{k=0}^{\infty} \left(\frac{9}{10} \right)^k = \frac{8}{1 - \frac{9}{10}} = 80, \text{ which is enough to conclude that the series converges absolutely}$$

since all of its terms are nonnegative. (Remark: the true value of this series is about 22.9; it converges extremely slowly.)

10. Easily f is smooth away from 0, and at $x = 0$, it can be shown that all derivatives of f exist, vanish, and are continuous at 0—from this, IV is false. It is true—away from 0 the Taylor series has positive radius of convergence since f is a composition of analytic functions there, and at 0 the Taylor series for f is identically 0, which converges everywhere. However, III is false—the Taylor series at 0 is identically 0, but f is nonvanishing away from 0. So the Taylor series does not converge to f in a neighborhood of 0.

11. I is true by Cauchy-Schwarz. More precisely, applying the standard Cauchy-Schwarz

$$\text{inequality in finite-dimensional spaces, we have } \sum_{n=0}^N |a_n b_n| \leq \left(\sum_{n=0}^N |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^N |b_n|^2 \right)^{\frac{1}{2}} \\ \leq \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{\frac{1}{2}} < \infty, \text{ making } \sum_{n=0}^{\infty} (a_n b_n) \text{ converge absolutely. It is false, as can be seen by}$$

$$\text{making } a_n = \frac{(-1)^{n+1}}{n+1}. \text{ III is false since we could make } a_n = -2^{n+1} \text{ and } b_n = \frac{1}{2 \cdot 2^{n+1}}, \text{ and in this}$$

case, $|a_n b_n| \not\rightarrow 0$, so $\sum_{n=0}^{\infty} (a_n b_n)$ has no chance of converging. (Remark: if we moreover specified that $a_n \geq 0$, this would have Dirichlet's test for convergence.)

12. Plugging in $f(x) = \sum_{n=0}^{\infty} (a_n x^n)$ into the differential equation, reindexing, and matching

coefficients, we find the recurrence $a_{n+1} = \frac{2^n}{n+1} a_n$, so by an easy induction, we find that

$$a_n = a_0 \prod_{i=0}^{n-1} \frac{2^i}{i+1} = \frac{2^{\frac{1}{2}(n^2-n)}}{n!}. \text{ Cauchy-Hadamard states that the radius of convergence } R \text{ for a}$$

power series $\sum_{n=0}^{\infty} (a_n y^n)$ satisfies $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$, with $R = 0$ if the limit on the right is infinite, or

$R = \infty$ if the limit on the right is 0; using this gives $\frac{1}{R} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}(n-1)}}{(n!)^{\frac{1}{n}}}$. By Stirling, $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$,

so this limit becomes $\frac{1}{R} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}(n-1)}}{(n!)^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}(n-1)}}{\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}(n-1)}}{\frac{1}{e} (\sqrt{2\pi})^{\frac{1}{n}} n^{1+\frac{2}{n}}} = \infty$, so $R = 0$.

13. The generic term of the sequence does not vanish, and $\frac{a_{n+2}}{a_{n+1}} = 3 - \frac{2a_n}{a_{n+1}}$. Passing to the limit

(assuming it does—but one can easily show that the sequence $\left\{\frac{a_{n+1}}{a_n}\right\}_{n \geq 0}$ is Cauchy and thus has

a limit), we find for the limit L that $L = 3 - \frac{2}{L}$, and the only solutions to this equation are $L = 1$ or $L = 2$; checking a few terms, one can intuit that the limit is 2. (Remark: alternatively, an easy induction shows that $a_n = 2^n - 1$, from which this limit also follows.)

14. No nontrivial such partition exists: the only one is the trivial partition into one arithmetic sequence with $a_0 = d = 1$. Suppose we've partitioned the positive integers as

$\bigcup_{j=1}^n \{a_0^{(j)} + kd_j \mid k \text{ is a nonnegative integer}\}$, and without loss of generality, $d_1 < d_2 < \dots < d_n$.

Consider the geometric series $\frac{z}{1-z} = \sum_{m=1}^{\infty} (z^m)$ as a power series in $\mathbb{C}$, which is convergent for $|z| < 1$ but remains bounded in a neighborhood of each point on the unit circle except $z = 1$.

Rearranging the series using our partition, we find $\frac{z}{1-z} = \sum_{j=1}^n z^{a_0^{(j)}} \sum_{n=0}^{\infty} (z^{d_j})^n = \sum_{j=1}^n \frac{z^{a_0^{(j)}}}{1-z^{d_j}}$ when

$|z| < 1$. Since $n \geq 2$, these $d_j > 1$; in particular, $d_n > 1$. Consider this identity near the point

$e^{\frac{2\pi i}{d_n}} \neq 1$; the left-hand side stays bounded while the right-hand side explodes. Hence, no such partition exists.

15. Starting with $a_0 = 1$, we find that a_n is precisely the number which, in decimal, is a string of 2^n ones. So the next term is a string of 16 ones, which is 1111111111111111.

16. The n th term is the squarefree part of n ; equivalently, a_n is what remains under the radical after fully simplifying $\sqrt{n}$. Therefore, $a_{18} = 2$ since $18 = 3^2 \cdot 2$.

17. This is the Collatz sequence starting from 27. More precisely, $a_{n+1} = \begin{cases} \frac{1}{2}a_n, & \text{if } a_n \text{ is even} \\ 3a_n + 1, & \text{if } a_n \text{ is odd} \end{cases}$.

So the next term is $3 \cdot 71 + 1 = 214$.

18. If we take the sequence of all nonnegative integers, written in increasing order, and change them to binary, the n th term of the given sequence is the number of ones in this binary representation. So the next term is $33 = 100001_2$, which contains 2 ones.

19. Write the integers starting with 0 in English: zero, one, two, three, four, five, six, seven, eight, nine, ten. The corresponding integers in the sequence is the number of “holes” (loops) in the English words. For example, e and o in zero and one each have a hole, o in two has a hole, each e in three has a hole, etc. The next number in English is eleven, and each e has a hole, so the next number in the sequence is 3.

20. Let X denote your payout. Then $X = 1.5^j$ when precisely j heads were flipped, which occurs with probability $\frac{1}{2^{j+1}}$. So $E(X) = \sum_{j=0}^{\infty} \frac{1.5^j}{2^{j+1}} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j = \frac{1}{2 \left(1 - \frac{3}{4}\right)} = 2$. Therefore, you’re willing

to pay at most \$2.

21. Let X denote your payout. To compute $E(\ln X)$, we note that the probabilities are the same as in the previous question; it’s just the values that have changed. Specifically, $E(\ln X)$

$= \sum_{j=0}^{\infty} \left(\frac{1}{2^{j+1}} \ln(2^j) \right) = (\ln 2) \sum_{j=0}^{\infty} \left(\frac{j}{2^{j+1}} \right)$. This series is hypergeometric, so setting S equal to this

sum, we have $S = \frac{0}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \dots$. Multiplying this by $\frac{1}{2}$ and subtracting that from the

original, we have $\frac{1}{2}S = \frac{0}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \dots \Rightarrow \frac{1}{2}S = S - \frac{1}{2}S = \left(\frac{0}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \dots \right)$

$-\left(\frac{0}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \dots \right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2} \Rightarrow S = 1$. Therefore, $E(\ln X) = 1 \ln 2$,

making $\$ \ln 2$ the maximum amount of money you are willing to pay.

22. The key insight here is to write $X = \sum_{j=1}^{42} (I_j)$, where I_j is 0 if restaurant j has not been visited

within k days and 1 if it has. It follows easily that $E(X) = \sum_{i=1}^{42} \left(1 - \left(1 - \frac{1}{42} \right)^k \right) = 42 \left(1 - \left(1 - \frac{1}{42} \right)^k \right)$.

$$23. \sum_{j=1}^{\infty} \left(\frac{2}{2^j + 2^{j-1}} \right) = \sum_{j=1}^{\infty} \left(\frac{2}{2^j \left(1 + \frac{1}{2} \right)} \right) = \sum_{j=1}^{\infty} \left(\frac{2}{2^j \left(\frac{3}{2} \right)} \right) = \frac{4}{3} \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \right) = \frac{4}{3} \cdot \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{4}{3}$$

$$24. \text{ Using the previous problem as a guide, } \sum_{j=1}^{\infty} \left(\frac{2}{\sum_{i=0}^{\infty} (2^{j-i})} \right) = \sum_{j=1}^{\infty} \left(\frac{2}{2^j \sum_{i=0}^{\infty} (2^{-i})} \right) = \sum_{j=1}^{\infty} \left(\frac{2}{2^j \cdot \frac{1}{1 - \frac{1}{2}}} \right)$$

$$= \sum_{j=1}^{\infty} \left(\frac{2}{2^j \cdot 2} \right) = \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \right) = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

$$\begin{aligned} 25. & \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(\frac{1}{2^{i+2j+3k}} \right) \right) \right) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(\frac{1}{2^i \cdot 4^j \cdot 8^k} \right) \right) \right) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\frac{1}{2^i \cdot 4^j} \sum_{k=1}^{\infty} \left(\frac{1}{8^k} \right) \right) \right) \\ & = \sum_{k=1}^{\infty} \left(\frac{1}{8^k} \right) \cdot \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\frac{1}{2^i \cdot 4^j} \right) \right) = \sum_{k=1}^{\infty} \left(\frac{1}{8^k} \right) \cdot \sum_{i=1}^{\infty} \left(\frac{1}{2^i} \sum_{j=1}^{\infty} \left(\frac{1}{4^j} \right) \right) = \sum_{k=1}^{\infty} \left(\frac{1}{8^k} \right) \cdot \sum_{j=1}^{\infty} \left(\frac{1}{4^j} \right) \cdot \sum_{i=1}^{\infty} \left(\frac{1}{2^i} \right) \\ & = \frac{\frac{1}{8}}{1 - \frac{1}{8}} \cdot \frac{\frac{1}{4}}{1 - \frac{1}{4}} \cdot \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{7} \cdot \frac{1}{3} \cdot 1 = \frac{1}{21} \end{aligned}$$

$$\begin{aligned} 26. & \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \left(\frac{3}{2^{i+2j}} \right) \right) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \left(\frac{3}{2^i \cdot 4^j} \right) \right) = \sum_{i=1}^{\infty} \left(\frac{3}{2^i} \sum_{j=1}^i \left(\frac{1}{4^j} \right) \right) = \sum_{i=1}^{\infty} \left(\frac{3}{2^i} \left(\frac{\frac{1}{4} \left(1 - \left(\frac{1}{4} \right)^i \right)}{1 - \frac{1}{4}} \right) \right) \\ & = \sum_{i=1}^{\infty} \left(\frac{3}{2^i} \cdot \frac{1}{3} \left(1 - \left(\frac{1}{4} \right)^i \right) \right) = \sum_{i=1}^{\infty} \left(\left(\frac{1}{2} \right)^i - \left(\frac{1}{8} \right)^i \right) = \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{\frac{1}{8}}{1 - \frac{1}{8}} = 1 - \frac{1}{7} = \frac{6}{7} \end{aligned}$$

27. Make an initial guess that $a_n = r^n$. Then $a_n = \frac{a_{n-1} + 2a_{n-2}}{3} \Rightarrow 3a_n - a_{n-1} - 2a_{n-2} = 0$
 $\Rightarrow 3r^n - r^{n-1} - 2r^{n-2} = 0$, and since r is clearly not 0, dividing both sides of this equation by r^{n-2}
 gives $0 = 3r^2 - r - 2 = (3r+2)(r-1) \Rightarrow r = -\frac{2}{3}$ or $r = 1$. Therefore, because this is linear,
 $a_n = A\left(-\frac{2}{3}\right)^n + B(1)^n = A\left(-\frac{2}{3}\right)^n + B$. Plugging in the given values, $-\frac{13}{3} = -\frac{2}{3}A + B$ and
 $-\frac{19}{9} = \frac{4}{9}A + B$. Subtracting the first equation from the second gives $\frac{20}{9} = \frac{10}{9}A \Rightarrow A = 2$, and
 plugging this value back into either equation gives $B = -3$. Therefore, $a_n = 2\left(-\frac{2}{3}\right)^n - 3$, and
 $\lim_{n \rightarrow \infty} (a_n) = -3$.

28. First, recognize that we must have $-1 < a, b < 1$ in order for the series on the left of the equation to converge, and it was also given that $a, b \neq 0$, so we will not consider those values throughout the problem. Under those conditions, the equation becomes

$$\frac{\frac{a}{1-a}}{\frac{b}{1-b}} = \frac{\frac{a}{b}}{1-\frac{a}{b}} \Rightarrow \frac{a(1-b)}{b(1-a)} = \frac{a}{b-a}, \text{ and since } a \neq 0, \text{ this simplifies to } \frac{1-b}{b(1-a)} = \frac{1}{b-a}.$$

Cross-multiplying gives $b-a-b^2+ab=b-ab \Rightarrow 2ab-a=b^2 \Rightarrow a(2b-1)=b^2 \Rightarrow a = \frac{b^2}{2b-1}$. Now,

consider the function $y = f(x) = \frac{x^2}{2x-1}$ (so that $b \sim x$ and $a \sim y$). $y' = \frac{2x(x-1)}{(2x-1)^2}$, so by sign

analysis, the function increases on $(-\infty, 0)$, then decreases on $(0, \frac{1}{2})$; there is an asymptote at

$x = \frac{1}{2}$; then the function decreases on $(\frac{1}{2}, 1)$, then decreases on $(1, \infty)$. Since $f(0) = 0$ and

$f(1) = 1$, there are no y -values on this function between 0 and 1, so we can now restrict to

$-1 < a < 0$. If $a < 0$, then $b < \frac{1}{2}$; if $-1 < a$, then $\frac{b^2+2b-1}{2b-1} > 0$, then $-1-\sqrt{2} < b < -1+\sqrt{2}$ or

$b > \frac{1}{2}$ —the overlap in these intervals is $-1-\sqrt{2} < b < -1+\sqrt{2}$, but we must cut off this interval

on the low end at -1 , due to the first comment in this solution. Therefore, $-1 < b < -1+\sqrt{2}$,

(which does make $-1 < a < 0$). Now, the sum on the right side of the equation is $\frac{a}{b-a}$

$$= \frac{\frac{b^2}{2b-1}}{b - \frac{b^2}{2b-1}} = \frac{b}{b-1} = 1 + \frac{1}{b-1}, \text{ which is a decreasing value for all } b < 1; \text{ therefore, the greatest}$$

$$\text{sum is } \frac{-1}{(-1)-1} = \frac{1}{2} \text{ and the least sum is } \frac{-1+\sqrt{2}}{(-1+\sqrt{2})-1} = -\frac{\sqrt{2}}{2} \text{ (with the exception of 0, of}$$

$$\text{course). This makes the sought product } \frac{1}{2} \left(-\frac{\sqrt{2}}{2} \right) = -\frac{\sqrt{2}}{4}.$$

$$29. \text{ The sum of the first four terms is } \frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} = \frac{30-10+5-3}{60} = \frac{22}{60} = \frac{11}{30}. \text{ By the}$$

alternating series test estimate, this value differs from the exact value by less than the absolute value of the next term: i.e., $\left| \frac{1}{30} \right| = \frac{1}{30}$. Therefore, the sum falls in the interval

$$\left[\frac{11}{30} - \frac{1}{30}, \frac{11}{30} + \frac{1}{30} \right] = \left[\frac{10}{30}, \frac{12}{30} \right] = \left[\frac{1}{3}, \frac{2}{5} \right].$$

$$30. \text{ The sum of the first four terms is } \frac{1}{2} + \frac{1}{10} + \frac{1}{30} + \frac{1}{68} = \frac{510+102+34+15}{1020} = \frac{661}{1020}. \text{ For this}$$

$$\text{estimate, we will need } \int \left(\frac{1}{x^3+x} \right) dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx = \ln|x| - \frac{1}{2} \ln(x^2+1) + C = \ln \frac{|x|}{\sqrt{x^2+1}} + C.$$

By the integral test estimate, the actual sum fall somewhere in the interval

$$\left[\frac{661}{1020} + \int_5^\infty \left(\frac{1}{x^3+x} \right) dx, \frac{661}{1020} + \int_4^\infty \left(\frac{1}{x^3+x} \right) dx \right]. \int_5^\infty \left(\frac{1}{x^3+x} \right) dx = \ln \frac{|x|}{\sqrt{x^2+1}} \Big|_5^\infty = 0 - \ln \frac{5}{\sqrt{26}}$$

$$= \ln \frac{\sqrt{26}}{5} \text{ and } \int_4^\infty \left(\frac{1}{x^3+x} \right) dx = \ln \frac{|x|}{\sqrt{x^2+1}} \Big|_4^\infty = 0 - \ln \frac{4}{\sqrt{17}} = \ln \frac{\sqrt{17}}{4}. \text{ Therefore, the sum is in the}$$

$$\text{interval } \left[\frac{661}{1020} + \ln \frac{\sqrt{26}}{5}, \frac{661}{1020} + \ln \frac{\sqrt{17}}{4} \right].$$