

## Answers:

1. B
2. A
3. C
4. E
5. A
6. D
7. B
8. A
9. D
10. E
11. B
12. C
13. D
14. C
15. C
16. A
17. A
18. D
19. B
20. C
21. B
22. C
23. B
24. D
25. A
26. B
27. E
28. A
29. D
30. C

Solutions:

1. From the second equation,  $y = x + 5$ , so substituting this into the first equation,

$$3x - (x + 5) = 13 \Rightarrow 2x - 5 = 13 \Rightarrow 2x = 18 \Rightarrow x = 9.$$

2. From the first two terms, the common difference is  $4 - (-2) = 6$ , so if  $n$  is the number of terms, then  $172 = -2 + 6(n - 1) \Rightarrow 174 = 6(n - 1) \Rightarrow 29 = n - 1 \Rightarrow 30 = n$ . Therefore, the sum of the series is  $\frac{30}{2}(-2 + 172) = 15(170) = 2550$ .

$$3. \frac{2x-1}{1+2x} = 3 \Rightarrow 2x-1 = 3(1+2x) = 3+6x \Rightarrow -4 = 4x \Rightarrow -1 = x$$

4. Adding the first and third equations' left sides and right sides yields  $x + y + z + w = x + z \Rightarrow w + y = 0$ . (As it turns out, the only solution to this system is  $x = y = z = w = 0$ .)

5. Let  $H$  and  $M$  be Henry's and Mary's current ages, respectively. The first sentence yields the equation  $\frac{H-20}{M-20} = \frac{2}{3} \Rightarrow 3H - 60 = 2(M - 20) = 2M - 40 \Rightarrow 3H = 2M + 20$

$H = \frac{2}{3}M + \frac{20}{3}$ . The second sentence yields the equation  $\frac{H+8}{M+8} = \frac{10}{13} \Rightarrow 13H + 104 = 10(M + 8)$

$10(M + 8) = 10M + 80$ , so plugging the first equation result into this second equation yields

$$\frac{26}{3}M + \frac{572}{3} = 13\left(\frac{2}{3}M + \frac{20}{3}\right) + 104 = 10M + 80 \Rightarrow \frac{332}{3} = \frac{4}{3}M \Rightarrow M = 83.$$

6. Let  $t_1$  and  $t_2$  be the times it takes for Mo to travel to and from T-Mobile Park, respectively,

and let  $d$  be the distance between Mo's house and T-Mobile Park. Then  $d = 50t_1 \Rightarrow t_1 = \frac{d}{50}$  and

$d = xt_2 \Rightarrow t_2 = \frac{d}{x}$ , where  $x$  is Mo's return speed. Therefore, the average speed over the entire

$$\text{round trip is } 60 = \frac{d+d}{\frac{d}{50} + \frac{d}{x}} = \frac{\cancel{d}(2)}{\cancel{d}\left(\frac{1}{50} + \frac{1}{x}\right)} = \frac{2}{\frac{x+50}{50x}} = \frac{100x}{x+50} \Rightarrow 60x + 3000 = 60(x + 50) = 100x$$

$$\Rightarrow 3000 = 40x \Rightarrow 75 = x.$$

$$7. \prod_{n=1}^7 \left( 1 + \frac{9}{n} + \frac{20}{n^2} \right) = \prod_{n=1}^7 \left( \frac{n^2 + 9n + 20}{n^2} \right) = \prod_{n=1}^7 \left( \frac{(n+4)(n+5)}{n^2} \right) = \frac{5 \cdot \cancel{6} \cdot \cancel{7} \cdot 8^2 \cdot 9^2 \cdot \cancel{10} \cdot 11^2 \cdot \cancel{12}}{1^2 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot \cancel{7}} \\ = 261,360$$

8.  $\frac{x_i^3}{1-3x_i+3x_i^2} = \frac{x_i^3}{(1-x_i)^3+x_i^3}$ , so taking two values of  $x_i$  that sum to 1 (i.e., for indices that sum to 101),

$$\frac{x_i^3}{(1-x_i)^3+x_i^3} + \frac{(1-x_i)^3}{(1-(1-x_i))^3+(1-x_i)^3} = \frac{x_i^3}{(1-x_i)^3+x_i^3} + \frac{(1-x_i)^3}{x_i^3+(1-x_i)^3} = \frac{(1-x_i)^3+x_i^3}{(1-x_i)^3+x_i^3} = 1. \text{ Since}$$

there are 51 such pairs (0-101, 1-100, 2-99, ..., 50-51), the sum is 51.

9. You could solve this problem by setting  $3x^2 + x = 23$ , solving for  $x$ , then plugging that number into the expression to the right of the equal sign, but the solutions are irrational and not easy to work with. Instead,  $f(3x^2 + x) = 9x^4 + 6x^3 + 7x^2 + 2x + 1 = (3x^2 + x)^2 + 2(3x^2 + x) + 1 \Rightarrow f(z) = z^2 + 2z + 1 = (z+1)^2 \Rightarrow f(23) = (23+1)^2 = 24^2 = 576$ .

10. Let  $x$  be the number of students in my lazy class. Then  $\frac{1}{6}x + 3 = \frac{1}{4}x$ . Multiplying both sides of this equation by 12 yields  $2x + 36 = 3x \Rightarrow 36 = x$ .

11. Since the trace is the sum of the entries on the main diagonal, we do not have to find the entire product, just the parts of the product that fall on the main diagonal:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 7 & 6 \end{bmatrix} \begin{bmatrix} -5 & 2 & 3 \\ 0 & -4 & 7 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 5(-5)+0+0 & & \\ & 2^2+4(-4)+0 & \\ & & 3^2+7^2+6(-6) \end{bmatrix} = \begin{bmatrix} -25 & & \\ & -12 & \\ & & 22 \end{bmatrix} \\ \Rightarrow (-25) + (-12) + 22 = -15.$$

12. Each factorial in the metafactorial ends in a number of 0s corresponding to the number of factors of 10 that are in the factorial, but factors of 10 are limited by the number of factors of 5 (there are fewer factors of 5 than factors of 2 in each). The factorials for 1, 2, 3, and 4 have zero 5 factors; for 5, 6, 7, 8, and 9, there is one factor of 5 in each; for 10, 11, 12, 13, and 14, there are two factors of 5 in each; for 15, 16, 17, 18, and 19, there are three factors of 5 in each; for 20, 21, 22, 23, and 24, there are four factors of 5 in each; for 25, 26, 27, 28, and 29, there are six factors of 5 in each; for 30, 31, 32, 33, and 34, there are seven factors of 5 in each;

and for 35, 36, and 37, there are eight factors of 5 in each. Therefore, the number of factors of 5 is  $4(0) + 5(1+2+3+4+6+7) + 3(8) = 139$ .

13. Another root must be  $4i$ , and the quadratic with  $4i$  and  $-4i$  as its roots is  $x^2 + 16$ .

Therefore, dividing yields  $f(x) = x^4 - 4x^3 - 5x^2 - 64x - 336 = (x^2 + 16)(x^2 - 4x - 21)$

$$(x^2 + 16)(x - 7)(x + 3), \text{ making the real roots } 7 \text{ and } -3. \text{ Therefore, } \frac{a^2 + b^2}{2ab} = \frac{7^2 + (-3)^2}{2(7)(-3)}$$

$$= \frac{49 + 9}{-42} = -\frac{58}{42} = -\frac{29}{21}.$$

14. Using the quadratic formula,  $x = \frac{1+i \pm \sqrt{(1-i)^2 - 4(1)(-6+2i)}}{2(1)} = \frac{1+i \pm \sqrt{24-10i}}{2}$

$$= \frac{1+i \pm (5-i)}{2} = 3-i \text{ or } -2, \text{ so the imaginary solution is } 3-i. \text{ (To see the simplification of the}$$

square root, one could use the method of finding complex roots that uses DeMoivre's theorem:

$$x^2 = 24 - 10i = 26 \left( \frac{12}{13} - \frac{5}{13}i \right) = 26 \operatorname{cis} \left( 2\pi - \cos^{-1} \frac{12}{13} \right) \Rightarrow x = \sqrt{26} \operatorname{cis} \left( \pi - \frac{1}{2} \cos^{-1} \frac{12}{13} \right) \text{ (only one}$$

answer is necessary since the answers to this are positive and negative of the same quantity,

$$\text{which is covered by the } \pm \text{ in the quadratic formula) } = \sqrt{26} \left( -\cos \left( \frac{1}{2} \cos^{-1} \frac{12}{13} \right) \right.$$

$$\left. + i \sin \left( \frac{1}{2} \cos^{-1} \frac{12}{13} \right) \right) = \sqrt{26} \left( -\sqrt{\frac{1+\frac{12}{13}}{2}} + i \sqrt{\frac{1-\frac{12}{13}}{2}} \right) = \sqrt{26} \left( -\frac{5}{\sqrt{26}} + \frac{1}{\sqrt{26}}i \right) = -5 + i, \text{ which is the}$$

negative of what is written as the result above.)

15. The angle of rotation  $\theta$  satisfies  $\tan 2\theta = \frac{110\sqrt{3}}{201-91} = \sqrt{3} \Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$ , so making the

$$\text{substitutions } x = (\cos 30^\circ)x' - (\sin 30^\circ)y' = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \text{ and } y = (\sin 30^\circ)x' + (\cos 30^\circ)y'$$

$$= \frac{1}{2}x' + \frac{\sqrt{3}}{2}y', \text{ we get } 2304 = 201 \left( \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \right)^2 + 110\sqrt{3} \left( \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \right) \left( \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \right)$$

$$+ 91 \left( \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \right)^2 = \frac{603}{4}(x')^2 - \frac{201\sqrt{3}}{2}x'y' + \frac{201}{4}(y')^2 + \frac{330}{4}(x')^2 + \frac{165\sqrt{3}}{2}x'y' - \frac{55\sqrt{3}}{2}x'y'$$

$-\frac{330}{4}(y')^2 + \frac{91}{4}(x')^2 + \frac{91\sqrt{3}}{2}x'y' + \frac{273}{4}(y')^2 = 256(x')^2 + 36(y')^2 \Rightarrow 1 = \frac{(x')^2}{9} + \frac{(y')^2}{64}$ , so we can see that before the rotation, this was an ellipse centered at the origin with half the major axis, the length from center to vertex, equal to  $\sqrt{64} = 8$ . Since the question is asking for distance from a vertex to the origin (center), this is 8.

16. Since 3 and 7 are not both powers of the same integer, in order to solve the equation, the exponents must both be 0. Setting the two exponents equal to each other, we get  $2x^3 - x^2 - 13x - 6 = 2x^3 - 5x^2 - 23x - 10 \Rightarrow 0 = 4x^2 + 10x + 4 = 2(2x+1)(x+2)$ , so the only integer that makes the exponents equal is  $-2$ . Now, to verify that this value makes the exponents 0, simply plug into one of them:  $2\left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 - 13\left(-\frac{1}{2}\right) - 6 = -\frac{1}{4} - \frac{1}{4} + \frac{13}{2} - 6 = 0$ . Therefore, the sum of the integral values of  $x$  is  $-2$ .

17. Suppose that this sequence had at least three terms. Because the sequence is arithmetic, we could call the first three terms  $a-d$ ,  $a$ , and  $a+d$ , where  $d$  is the common difference. Then, because the sequence is also geometric, we have  $\frac{a}{a-d} = \frac{a+d}{a} \Rightarrow a^2 - d^2 = a^2 \Rightarrow d = 0$ , which makes all three terms equal to  $a$ , a contradiction that not all of the terms in the sequence were equal. Therefore, the maximum number of terms of such a sequence is 2 (for example, if the sequence is  $x, y$ , then the common difference of the sequence is  $y-x$  and the common ratio is  $\frac{y}{x}$  (since  $x > 0$ , this ratio is acceptable).

18. Since  $0 < \arcsin\left(\frac{\sqrt{6}}{3}\right), \arcsin\left(\frac{1}{3}\right) < \frac{\pi}{2}$  and  $\arcsin\left(\frac{\sqrt{6}}{3}\right) > \arcsin\left(\frac{1}{3}\right)$ , it must be the case that  $0 < 2\arcsin\left(\frac{\sqrt{6}}{3}\right) - \arcsin\left(\frac{1}{3}\right) < \pi$ . Therefore, because  $\sin\left(2\arcsin\left(\frac{\sqrt{6}}{3}\right) - \arcsin\left(\frac{1}{3}\right)\right)$

$$= \sin\left(2\arcsin\left(\frac{\sqrt{6}}{3}\right)\right)\cos\left(\arcsin\left(\frac{1}{3}\right)\right) - \cos\left(2\arcsin\left(\frac{\sqrt{6}}{3}\right)\right)\sin\left(\arcsin\left(\frac{1}{3}\right)\right)$$

$$= 2\sin\left(\arcsin\left(\frac{\sqrt{6}}{3}\right)\right)\cos\left(\arcsin\left(\frac{\sqrt{6}}{3}\right)\right)\cos\left(\arcsin\left(\frac{1}{3}\right)\right) - \left(1 - 2\sin^2\left(\arcsin\left(\frac{\sqrt{6}}{3}\right)\right)\right)\sin\left(\arcsin\left(\frac{1}{3}\right)\right)$$

$$\sin\left(\arcsin\left(\frac{1}{3}\right)\right) = 2\left(\frac{\sqrt{6}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(1 - 2\left(\frac{\sqrt{6}}{3}\right)^2\right)\left(\frac{1}{3}\right) = \frac{8}{9} - \left(-\frac{1}{9}\right) = 1, \text{ we must have}$$

$$2\arcsin\left(\frac{\sqrt{6}}{3}\right) - \arcsin\left(\frac{1}{3}\right) = \frac{\pi}{2}.$$

19. The standard equation of the circle is  $(x^2 - 4x + 4) + (y^2 + 2y + 1) = 15 + 4 + 1$

$\Rightarrow (x-2)^2 + (y+1)^2 = 20$ , meaning the center of the circle is  $(2, -1)$ . Since the center is also on the given line, the points of intersection are endpoints of a diameter of the circle. Therefore, the distance between them is  $2\sqrt{20} = 4\sqrt{5}$ .

20. Because  $S_n$  is the sum of the first  $n$  positive prime numbers,  $\sum_{n=1}^{10} S_n$  will be the sum ten copies of the least prime, nine copies of the next least prime, et cetera, down to one copy of the tenth least prime. Therefore,  $\sum_{n=1}^{10} S_n = 10(2) + 9(3) + 8(5) + 7(7) + 6(11) + 5(13) + 4(17) + 3(19) + 2(23) + 1(29) = 467$ .

21. Using the fact that  $\binom{n}{k} = \binom{n}{n-k}$ ,  $\binom{2023}{1368} + \binom{2023}{654} + \binom{2024}{1370} + \binom{2025}{654} = \binom{2023}{1368} + \binom{2023}{1369} + \binom{2024}{1370} + \binom{2025}{1371}$ . Now, because of the recursive definition of Pascal's Triangle  $\left(\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}\right)$ ,  $\binom{2023}{1368} + \binom{2023}{1369} + \binom{2024}{1370} + \binom{2025}{1371} = \binom{2024}{1369} + \binom{2024}{1370} + \binom{2025}{1371} = \binom{2025}{1370} + \binom{2025}{1371} = \binom{2026}{1371}$ . Since this is also equal to  $\binom{2026}{655}$ , the greater value of  $n$  is 1371.

22. Write  $AB8 = 100A + 10B + 8$ ,  $2C = 20 + C$ , and  $5DE6 = 5000 + 100D + 10E + 6$ , meaning that  $5000 + 100D + 10E + 6 = (100A + 10B + 8)(20 + C) = 2000A + 200B + 160 + 100AC + 10BC + 8C$ ; the only terms that contributes to the units digit are 6 on the left and  $8C$  on the right, and of the given numbers, the only one that when multiplied by 8 ends in a 6 is 7, so  $C = 7$ . Plugging this into the above equation yields  $5000 + 100D + 10E + 6 = 2000A + 200B + 160 + 700A + 70B + 56 \Rightarrow 4790 + 100D + 10E = 2700A + 270B \Rightarrow 479 + 10D + E = 270A + 27B$ . The right side of this equation is divisible by 9, so  $479 + 10D + E \equiv 0 \pmod{9} \Rightarrow 2 + D + E \equiv 0 \pmod{9}$

$\Rightarrow D + E \equiv 7 \pmod{9}$ . Of the remaining digits (all of the digits are distinct), only 3 and 4 fit this description, so  $D$  and  $E$  are 3 and 4, in some order; this also means that  $A$  and  $B$  are 1 and 9, in some order. If  $A = 9$ , then  $AB8 > 900$  and  $2C > 20$ , then  $(AB8)(2C) > 900 \cdot 20 = 18,000$ , so this would not give the appropriate product since it is a four-digit number—therefore,  $A = 1$  and  $B = 9$ . Plugging this into our simplified equation,  $479 + 10D + E = 270(1) + 27(9) = 513$   
 $\Rightarrow 10D + E = 34 \Rightarrow D = 3$  and  $E = 4$ . Therefore,  $5A + 4B + 3C + 2D + E = 5(1) + 4(9) + 3(7) + 2(3) + 4 = 72$ .

23. Both  $(1, 2)$  and its reflection will be on the same line perpendicular to  $y = 2x - 1$ , so the slope of that line is  $-\frac{1}{2}$ . Since  $(1, 2)$  is on this line, its equation is  $y - 2 = -\frac{1}{2}(x - 1)$   
 $\Rightarrow y = -\frac{1}{2}x + \frac{5}{2}$ , and the intersection of these two lines satisfies  $-\frac{1}{2}x + \frac{5}{2} = 2x - 1 \Rightarrow \frac{5}{2}x = \frac{7}{2}$   
 $\Rightarrow x = \frac{7}{5} \Rightarrow y = \frac{9}{5}$ , so the intersection is  $\left(\frac{7}{5}, \frac{9}{5}\right)$ . To get from  $(1, 2)$  to  $\left(\frac{7}{5}, \frac{9}{5}\right)$ ,  $x$  increases by  $\frac{2}{5}$  while  $y$  decreases by  $\frac{1}{5}$ . If we make these changes to  $\left(\frac{7}{5}, \frac{9}{5}\right)$ , we get  $\left(\frac{7}{5} + \frac{2}{5}, \frac{9}{5} - \frac{1}{5}\right) = \left(\frac{9}{5}, \frac{8}{5}\right)$ , making this the reflection point.

24. By Euler's Theorem, since 17 and 1000 are relatively prime,  $17^{\phi(1000)} \equiv 1 \pmod{1000}$   
 $17^{400} = 17^{1000 \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{2}\right)} \equiv 1 \pmod{1000} \Rightarrow 17^{2000} = (17^{400})^5 \equiv 1^5 = 1 \pmod{1000} \Rightarrow 17^{2003} = 17^{2000} \cdot 17^3$   
 $\equiv 1 \cdot 17^3 = 17^3 \pmod{1000}$ . Therefore,  $17^{2003}$  and  $17^3$  end in the same sequence of three digits. Since  $17^3 = 4913$ , the hundreds digit of  $17^{2003}$  is 9.

25. For the four-person jury to find for the correct verdict, three or four of the members must elect the correct verdict. Because the members elect independently, if exactly three members do so, this probability is  $\binom{4}{1} p^3 (1-p)^1 = 4p^3 - 4p^4$ ; if all four members do so, this probability is  $\binom{4}{0} p^4 (1-p)^0 = p^4$ . Therefore, for the one- and four-person juries to elect the correct verdict,  
 $p = (4p^3 - 4p^4) + p^4 \Rightarrow 0 = 3p^4 - 4p^3 + p = p(p-1)(3p^2 - p - 1)$ . Since  $0 < p < 1$ , we must have

$3p^2 - p - 1 = 0 \Rightarrow p = \frac{1 \pm \sqrt{1^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} = \frac{1 \pm \sqrt{13}}{6}$ . Since  $\frac{1 - \sqrt{13}}{6} < 0$ , it is not a valid probability; therefore,  $p = \frac{1 + \sqrt{13}}{6}$ .

26. Because  $e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ ,  $1.0001^{10000} = \left(1 + \frac{1}{10000}\right)^{10000} \approx e$ , and the closest integer to  $e$  is 3.

(Note: Just to show this, using a calculator,  $1.0001^{10000}$  has a decimal beginning 2.7181459....)

27.  $(\log x)^n = \log(x^n) \Rightarrow 0 = (\log x)^n - \log(x^n) = (\log x)^n - n \cdot \log x = (\log x)((\log x)^{n-1} - n)$

$\log x = 0$  or  $(\log x)^{n-1} - n = 0 \Rightarrow x = 1$  or  $x = 10^{\pm \sqrt[n-1]{n}}$  are the only two solutions, but this is only because since  $n$  is even,  $n-1$  is odd—if  $n-1$  had been even, the solutions to the second equation would have been  $x = 10^{\pm \sqrt[n-1]{n}}$ , giving three solutions.

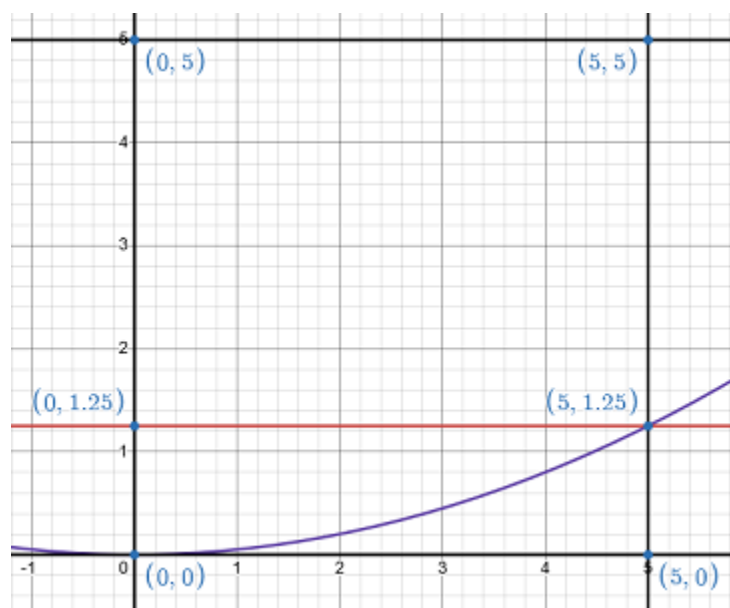
28. Let  $a$ ,  $b$ , and  $c$  be the roots of a cubic polynomial  $x^3 + Mx^2 + Nx + P$ , and by Vieta's formula, we are looking for the positive value of  $-M$ ; i.e., the value of  $M$  when it is negated. Further by Vieta's formula with the product of the roots,  $-P = 15 \Rightarrow P = -15$ . Now, setting  $S_n = a^n + b^n + c^n$

(hence,  $S_1 = -M$ ), using Newton's sums,  $0 = S_2 + M \cdot S_1 + N \cdot 2 = 15 - M^2 + 2N \Rightarrow N = \frac{M^2 - 15}{2}$ ;

additionally, using Newton's sums,  $0 = S_3 + M \cdot S_2 + N \cdot S_1 - 15 \cdot 3 = 31 + 15M - M \left( \frac{M^2 - 15}{2} \right) - 45$

$= -14 + \frac{45}{2}M - \frac{1}{2}M^3 \Rightarrow 0 = M^3 - 45M + 28 = (M+7)(M^2 - 7M + 4)$ , and the two solutions to the quadratic are positive. Therefore,  $M = -7 \Rightarrow -M = 7$ .

29. In order for the quadratic to have one or two real solutions, it must be the case that the discriminant is not negative; i.e.,  $b^2 - 20c = b^2 - 4 \cdot 5 \cdot c \geq 0 \Rightarrow c \leq \frac{1}{20}b^2$ . Sketching the graph of this, using  $b$  as the horizontal axis and  $c$  as the vertical axis, looks like the following:



The additional horizontal line in the diagram,  $c = 1.25$ , is sketched because that is where the parabolic arc intersects  $b = 5$ . Now, because we want  $c \leq \frac{1}{20}b^2$ , we are looking for the area inside the  $5 \times 5$  square that is below the parabolic arc. Using the information given in the problem, the area of the region above the parabolic arc but below the horizontal line  $c = 1.25$  is  $\frac{1}{2} \cdot \frac{2}{3} \cdot 10 \cdot \frac{5}{4} = \frac{25}{6}$  (the  $\frac{1}{2}$  is due to the fact that we are not counting the area to the left of  $b = 0$ ). The area above  $c = 1.25$  and below  $c = 5$  is  $5 \cdot \left(5 - \frac{5}{4}\right) = \frac{75}{4}$ , so the area of the sought region is  $5^2 - \frac{25}{6} - \frac{75}{4} = \frac{300 - 50 - 225}{12} = \frac{25}{12}$ . Therefore, the probability of selecting a point in this region is  $\frac{\frac{25}{12}}{25} = \frac{1}{12}$ .

30. Let  $f$  be a function defined on positive integers such that  $f(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$ ;

therefore,  $f$  maps positive integer  $n$  to the  $n$ th triangular number. To find the inverse of this function, solve this for positive  $n$  in the following way:  $2f(n) = n(n+1) = n^2 + n \Rightarrow 2f(n)$

$$+ 0.25 = n^2 + n + 0.25 = (n + 0.5)^2 \Rightarrow \sqrt{2f(n) + 0.25} - 0.5 = n \Rightarrow f^{-1}(n) = \sqrt{2n + 0.25} - 0.5.$$

Therefore, the expression inside the least integer function maps triangular numbers to the position of the triangular number:  $f^{-1}(1) = 1, f^{-1}(3) = 2, f^{-1}(6) = 3, f^{-1}(10) = 4$ , et cetera. Since the expression inside the least integer function is also increasing, this means that

$1 < f^{-1}(2) < 2$ ,  $2 < f^{-1}(4), f^{-1}(5) < 3$ ,  $3 < f^{-1}(7), f^{-1}(8), f^{-1}(9) < 4$ , et cetera. Since the least integer function rounds these numbers up, the series is  $1 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + \dots + \underbrace{n + n + \dots + n}_{n \text{ copies}} + \dots$ . To determine how many of the last integer is being summed, we need to

find the closest triangular number less than 2026. Since  $\frac{63 \cdot 64}{2} = 2016$  and  $\frac{64 \cdot 65}{2} = 2080$ , the

series ends with 63 copies of 63 and 10 copies of 64. Therefore,  $\sum_{n=1}^{2026} \left\lceil \sqrt{2n + 0.25} - 0.5 \right\rceil$

$$= 1(1) + 2(2) + 3(3) + \dots + 63(63) + 10(64) = 1^2 + 2^2 + 3^2 + \dots + 63^2 + 640 = \frac{63 \cdot 64 \cdot 127}{6} + 640$$

$$= 85,984.$$