

Answers:

1. B
2. A
3. B
4. D
5. B
6. D
7. C
8. D
9. A
10. B
11. C
12. D
13. A
14. E
15. D
16. E
17. B
18. E
19. B
20. C
21. B
22. C
23. C
24. E
25. A
26. B
27. A
28. D
29. B
30. E

Solutions:

1. Since $4052 = 4 \cdot 1013$ and $9117 = 9 \cdot 1013$, the gcd is 1013.
2. Note that $9 \cdot 101 = 909 < 1001 < 1010 = 10 \cdot 101$. Hence, $q = 9 \Rightarrow r = 1001 - 909 = 92$
 $\Rightarrow q + r = 9 + 92 = 101$.
3. Since 2024 is the greatest multiple of 4 less than 2026, we have $2^{2024} \equiv 6 \pmod{10}$. Then
 $2^{2026} = 4 \cdot 2^{2024} \equiv 4 \cdot 6 = 24 \equiv 4 \pmod{10}$, so the units digit is 4.
4. The number of factors of 7 in $2401!$ is $\left\lfloor \frac{2401}{7} \right\rfloor + \left\lfloor \frac{2401}{49} \right\rfloor + \left\lfloor \frac{2401}{343} \right\rfloor + \left\lfloor \frac{2401}{2401} \right\rfloor = 343 + 49 + 7$
 $+1 = 400$.
5. $3x + 1 \equiv 0 \pmod{11} \Rightarrow 3x \equiv 10 \pmod{11} \Rightarrow 3x \equiv 21 \pmod{11} \Rightarrow x \equiv 7 \pmod{11}$, so $x = 7$ is the smallest positive integer solution to this equivalence.
6. Since $1001 = 7 \cdot 11 \cdot 13$, the sum of the positive integer divisors is $(1+7)(1+11)(1+13)$
 $= 8 \cdot 12 \cdot 14 = 1344$.
7. We are looking for the greatest prime less than 200 such that the integer 4 less and 6 less are also prime. Note that 199 is prime, but 195 is not. Next, 197 is prime, 193 is prime, and 191 is prime, so the sought prime trio is 191, 193, and 197. Their sum is $191 + 193 + 197 = 581$.
8. The digit sum of $28577d240$ is $35 + d$, so to be divisible by 3, d must be 1, 4, or 7. To be divisible by 7, $-285 + 77d - 240$ must be divisible by 7. Since this reduces to $77d - 525$, we try each of 1, 4, and 7. We find that if $d = 7$, $777 - 525 = 252 = 7 \cdot 36$, so 285777240 is divisible by both 3 and 7, making it divisible by 21. The other two options, 1 and 4, do not make the number $28577d240$ divisible by 7.
9. The four-digit integers with digit sum 2 are 1001, 1010, 1100, and 2000. Of these, only 1001 and 1010 are trippypriimes: $1001 = 7 \cdot 11 \cdot 13$ and $1010 = 2 \cdot 5 \cdot 101$. Then $s_1 = 7 + 11 + 13 = 31$ and $s_2 = 2 + 5 + 101 = 108$, making $|s_1 - s_2| = |31 - 108| = |-77| = 77$.

10. A factorization of $3^{16} - 2^{16}$ is $(3^8 + 2^8)(3^4 + 2^4)(3^2 + 2^2)(3 + 2)(3 - 2) = 6817 \cdot 97 \cdot 13 \cdot 5$. We can further factor 6817 as $17 \cdot 401$, so we now have the prime factorization. Of the answer choices, only 47 is not a prime factor.

11. Since $659 = 64 \cdot 10 + 19$, $\frac{659}{64} = 10 + \frac{19}{64} = 10 + \frac{76}{256} = 10 + \frac{64 + 12}{256} = 10 + \frac{64}{256} + \frac{12}{256} = 10 + \frac{4}{16} + \frac{12}{256}$, so the hexadecimal representation is $A.4C_{16}$.

12. The prime factorization of 11088 is $2^4 \cdot 3^2 \cdot 7 \cdot 11$, so we use Euler's totient function to calculate the number for which we are looking. $\phi(11088) = 2^4 \cdot 3^2 \cdot 7 \cdot 11 \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{6}{7}\right)\left(\frac{10}{11}\right) = 2^6 \cdot 3^2 \cdot 5 = 2880$.

13. We use the Euclidean algorithm to get the greatest common divisor of 132 and 509, then back-substitute: $509 = 132 \cdot 3 + 113$, $132 = 113 \cdot 1 + 19$, $113 = 19 \cdot 5 + 18$, $19 = 18 \cdot 1 + 1$. Now we have $1 = 19 - 18 = 19 - (113 - 19 \cdot 5) = 6 \cdot 19 - 113 = 6(132 - 113) - 113 = 6 \cdot 132 - 7 \cdot 113 = 6 \cdot 132 - 7(509 - 132 \cdot 3) = 27 \cdot 132 + (-7) \cdot 509$. Hence, one option for $|a + b|$ is $|27 - 7| = 20$. To get other solutions, we increase/decrease a by 509 some number of times while we decrease/increase b by 132 the same number of times—either of these options yields a larger sum than the 20 we already have.

14. Since $7 \cdot 14 = 98$, the smallest three-digit m is $m = 102 = 98 + 4 \equiv 4 \pmod{7}$; similarly, since $11 \cdot 91 = 1001$, the smallest four-digit n is $n = 1003 = 1001 + 2 \equiv 2 \pmod{11}$. Now we use the Euclidean algorithm: $1003 = 102 \cdot 9 + 85$, $102 = 85 \cdot 1 + 17$, $85 = 17 \cdot 5$. Therefore, $\gcd(102, 1003) = 17$.

15. Because $\gcd(21, 77) = 7$ divides 35, we can divide the congruence and the modulus by 7 to get $3x \equiv 5 \pmod{11} \Rightarrow 3x \equiv 27 \pmod{11} \Rightarrow x \equiv 9 \pmod{11}$. The integers that satisfy this equivalence are 9, 20, 31, 42, 53, 64, and 75, and the average of those numbers is 42.

16. Since $2026 \equiv 10 \pmod{21}$, we simplify the problem to $x \equiv 10^{30} \pmod{21}$. Note that since $\phi(21) = 21 \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = 12$, we have $10^{12} \equiv 1 \pmod{21}$ by Euler's Theorem. It follows that $10^{30} = (10^{12})^2 10^6 \equiv 1^2 10^6 = 10^6 \pmod{21}$. Now, since $100 \equiv -5 \pmod{21}$, we have

$10^6 = (10^2)^3 \equiv (-5)^3 = -125 \pmod{21}$, which means $10^6 \equiv -125 + 6 \cdot 21 = 1 \pmod{21}$. Therefore, the remainder is 1.

17. All solutions to $x \equiv 4 \pmod{11}$ are of the form $x = 4 + 11t$, where t is some integer. Hence, $4 + 11t \equiv 3 \pmod{17} \Rightarrow 11t \equiv -1 \pmod{17} \Rightarrow 11t \equiv 33 \pmod{17} \Rightarrow t \equiv 3 \pmod{17}$, which means that $t = 3 + 17s$ for some integer s . Therefore, $x = 4 + 11(3 + 17s) = 37 + 187s \Rightarrow x \equiv 37 \pmod{187}$, and this way of writing the solution has the lowest possible sum for $m + n = 187 + 37 = 224$, since 187 is fixed and x either increases by some number of 187s, increasing the sum, or decreases by some number of 187s, making it negative.

18. We solve this by inverting the problem. Instead of working with the interval $\left[\frac{11}{27}, \frac{11}{26}\right]$ and

finding a rational number $\frac{m}{n}$, we work with the interval $\left[\frac{26}{11}, \frac{27}{11}\right]$ and find a rational number

$\frac{n}{m}$, where n is a minimum. Therefore, since $\left[\frac{26}{11}, \frac{27}{11}\right] = \left[2 + \frac{4}{11}, 2 + \frac{5}{11}\right]$, we must have that

$2 + \frac{4}{11} < \frac{n}{m} < 2 + \frac{5}{11}$. Hence, $\frac{n}{m}$ can be written as $2 + \frac{x}{m}$, where x is some positive integer.

Then we have $\frac{4}{11} < \frac{x}{m} < \frac{5}{11}$. Clearing fractions, this becomes $4m < 11x < 5m$. The smallest

multiple of 11 strictly between the same integer multiple of 4 and 5 is 22; indeed, if $m = 5$ and

$x = 2$, then $20 < 22 < 25$. Since $m = 5$ and $x = 2$, our fraction $2 + \frac{x}{m}$ becomes $2 + \frac{2}{5} = \frac{12}{5}$.

Thus, the original fraction must be $\frac{5}{12}$, making the sum $m + n = 5 + 12 = 17$.

19. We are told that $1220 = 1 + p^2 + q^2 + (pq)^2 = 1 + p^2 + q^2 + p^2q^2 = (1 + p^2)(1 + q^2)$; hence, we require factors of 1220 that are one more than perfect squares. The only way to do this is $10 \cdot 122$, which makes p and q equal to 3 and 11. Therefore, $|p - q| = |11 - 3| = 8$.

20. We want $9k^2 - 8$ to be a perfect square, so we set $9k^2 - 8 = n^2 \Rightarrow 9k^2 - n^2 = 8$
 $\Rightarrow (3k - n)(3k + n) = 8$. Thus, since all variables are integers, the factors $3k - n$ and $3k + n$ must be factors of 8. Hence, we have the following possibilities:

$$\begin{array}{rcccccccc} 3k - n & = & -8 & -4 & -2 & -1 & 1 & 2 & 4 & 8 \\ 3k + n & = & -1 & -2 & -4 & -8 & 8 & 4 & 2 & 1 \end{array}$$

Adding equations in each case cancels n , and we obtain values of $6k$: $-9, -6, -6, -9, 9, 6, 6, 9$, respectively, and the only integer solutions are $k = \pm 1 \Rightarrow n = \pm 1$. Hence, these are the only two possible values of k .

21. Consider $n - m = (n + k) - (m + k)$. Since $m + k$ divides both $n + k$ and itself, it divides the difference $n - m$ for integers k from 1 to 10, inclusive. Since we have 10 consecutive integers $m + 1, m + 2, \dots, m + 10$, at least one of these is divisible by 11. Likewise, at least one of these 10 integers is divisible by 10, 9, 8, 7, and so on. Hence, $n - k$ is divisible by $11!$, which implies n is divisible by $11! + k$. However, we want the minimum value. Thus, the minimum value that $n - k$ can be divisible by must contain factors of $11!$ while being less than $11!$. Choosing $k = 1$ gives the minimum as $n = 11 \cdot 9 \cdot 8 \cdot 7 \cdot 5 + 1 = 27721$.

22. Squaring both sides of the equation gives $a + 2\sqrt{2026} = b + c + 2\sqrt{bc}$, so because a , b , and c are integers, we must have $bc = 2026$. The only factorizations of 2026 are $1 \cdot 2026$ and $2 \cdot 1013$, and since $a = b + c$, the minimum value of a is $2 + 1013 = 1015$.

23. Note that in mod 47, $42! \equiv 42! \cdot 48 = 42!(-4)(-3)(-2)(-1) \cdot 2 \equiv 42! \cdot 43 \cdot 44 \cdot 45 \cdot 46 \cdot 2 = 46! \cdot 2$. By Wilson's Theorem, $46! \equiv -1 \pmod{47}$, so $42! \equiv 46! \cdot 2 \equiv -1 \cdot 2 = -2 \equiv 45 \pmod{47}$, so the remainder is 45.

24. All of the answer choices are four-digit numbers. That said, $37^2 + 1$ ends in a 0, so it is divisible by 10 and not prime; $48^2 + 1$ and $88^2 + 1$ end in a 5, so they are divisible by 5 and not prime; and $79^2 + 1$ ends in a 2, so it is divisible by 2 and not prime. None of the answer choices are prime.

25. We wish to have $\phi(N) = 120$. Since $120 = 12 \cdot 10$, and the integers one greater than each of those factors is prime (11 and 13), $N = 11 \cdot 13 = 143$ would make $\phi(143) = 143 \cdot \left(\frac{10}{11}\right) \left(\frac{12}{13}\right) = 120$, and this is the smallest such N . Therefore, $N = 11 \cdot 13 \equiv -6 \cdot -4 = 24 \equiv 7 \pmod{17}$, making the remainder 7.

26. Since $120 = 2^3 \cdot 3 \cdot 5$, we seek a sum of powers of primes that sum to 8, 3, and 5; or 24 and 5; or 40 and 3; or 8 and 15; or 120. However, neither 5 nor 120 is a sum of powers of primes, so it must be 40 and 3 or 8 and 15. Checking, we get $8 \cdot 15 = (1 + 7)(1 + 2 + 2^2 + 2^3)$ and $3 \cdot 40$

$= (1+2)(1+3+3^2+3^3)$, which implies that either $n = 7 \cdot 2^3 = 56$ or $n = 2 \cdot 3^3 = 54$.

$54 \equiv 3 \pmod{17}$, so the remainder is 3.

27. One more than each of the exponents in the prime factorization of N must have a product of 120. Since the prime factorization is $120 = 2^3 \cdot 3 \cdot 5$, these exponents could be 4, 2, 1, 1, and 1; or 7, 4, and 2; or 23 and 4; or 39 and 2; or 14 and 7; or 119. Since we want the smallest such N , we make the largest exponent the power 2 and the next largest the power of 3. The smallest is clearly $N = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 16 \cdot 9 \cdot 5 \cdot 7 \cdot 11$, which in mod 17 satisfies $N \equiv -1 \cdot (-8) \cdot 5 \cdot 7 \cdot 11 = 40 \cdot 77 \equiv 6 \cdot 9 = 54 \equiv 3$, so the remainder is 3.

28. Using the formula for the sum of squares, we have $k = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$.

For k to be an integer, 6 must divide $(n+1)(2n+1)$, and since $2n+1$ will never be divisible by 2, there are two cases to consider: I) 2 divides $n+1$ and 3 divides $2n+1$, or II) 6 divides $n+1$.

In case I, if 2 divides $n+1$, then n is odd, implying there exists an integer m such that $n = 2m+1$. Therefore, $2n+1 = 4m+3$, which is only divisible by 3 when m is a multiple of 3; this in turn implies that n is one more than a multiple of 6. These are the numbers 1, 7, 13, ..., 2023, and there are $\frac{2023-1}{6} + 1 = 338$ of those. In case II, if 6 divides $n+1$, then $n \equiv 5 \pmod{6}$,

which is the integers 5, 11, 17, ..., 2021, and there are $\frac{2021-5}{6} + 1 = 337$ of those. Altogether, the total number of values of n that make k an integer is $338 + 337 = 675$.

29. Note that for $n > m$, $2^n - 1$ can be written in terms of $2^m - 1$ in the following way:

$2^n - 1 = (2^m - 1)(2^{n-m} + 2^{n-2m} + \dots + 2^{n(\text{mod } m)} + 1) + 2^{n(\text{mod } m)} - 1$. We will find the greatest common divisor by using this fact, along with the Euclidean Algorithm:

$$\begin{aligned} 2^{2206} - 1 &= (2^{2026} - 1)(2^{180}) + (2^{180} - 1) \\ 2^{2026} - 1 &= (2^{180} - 1)(2^{1846} + 2^{1666} + 2^{1486} + \dots + 2^{46}) + (2^{46} - 1) \\ 2^{180} - 1 &= (2^{46} - 1)(2^{134} + 2^{88} + 2^{42}) + (2^{42} - 1) \\ 2^{46} - 1 &= (2^{42} - 1)(2^4) + (2^4 - 1) \\ 2^{42} - 1 &= (2^4 - 1)(2^{38} + 2^{34} + \dots + 2^2) + (2^2 - 1) \\ 2^4 - 1 &= (2^2 - 1)(2^2 + 1) \end{aligned}$$

We have reached a point where the remainder is 0, so the gcd is the previous nonzero remainder, which is $2^2 - 1 = 3$.

(Note: This is indicative of a more general property for relatively prime positive integers a and b , and positive integers m and n with $m < n$: $\gcd(a^m - b^m, a^n - b^n) = a^{\gcd(m,n)} - b^{\gcd(m,n)}$. See p. 147 in *Concrete Mathematics*, second edition, by Knuth, Patashnik, & Graham.)

30. Let $u = x + y$ and $v = xy$. Then $x^3 + y^3 = (x + y)^3 - 3xy(x + y) = u^3 - 3uv$, $x^2y + xy^2 = xy(x + y) = uv$, and $x^2 + xy + y^2 = (x + y)^2 - xy = u^2 - v$. Plugging these in, the equation becomes $u^3 - 2uv = 8(u^2 - v + 1)$, or equivalently, $u(u^2 - 2v) = 8(u^2 - v + 1)$. It follows that u is even, so let $u = 2w$. Then $2w(4w^2 - 2v) = 8(4w^2 - v + 1) \Rightarrow w(2w^2 - v) = 2(4w^2 - v + 1)$

$$2w^3 - vw = 8w^2 - 2v + 2 \Rightarrow 2w^3 - 8w^2 - 2 = vw - 2v = v(w - 2) \Rightarrow v = \frac{2w^3 - 8w^2 - 2}{w - 2}$$

$$= 2w^2 - 4w - 8 - \frac{18}{w - 2}.$$

Hence $w - 2$ must divide 18, implying w is 0, 1, 3, 5, 8, 11, or 20; the corresponding values of v are 1, 8, -20, 16, 85, 188, or 711. To determine which of these is correct without tedious computation, we note that $w^2 - v = \frac{u^2}{4} - v = \frac{x^2 + 2xy + y^2}{4} - xy$

$$= \frac{x^2 - 2xy + y^2}{4} = \left(\frac{x - y}{2}\right)^2,$$

so $w^2 - v$ is a perfect square. The only pair that is a perfect square is $w^2 - v = 5^2 - 16 = 9$. Therefore, $x + y = u = 2w = 2 \cdot 5 = 10$. (By the way, this means that there are two solutions: $(x, y) = (2, 8)$ or $(x, y) = (8, 2)$.)