

Answers:

1. B
2. A
3. A
4. B
5. B
6. C
7. C
8. B
9. E
10. D
11. B
12. A
13. E
14. C
15. D
16. E
17. A
18. C
19. A
20. C
21. E
22. B
23. D
24. B
25. B
26. C
27. C
28. D
29. D
30. A

Solutions:

1. There are three prime numbers (2, 3, and 5) on the die, so the probability is $\frac{3}{6} = \frac{1}{2}$.

2. There are two each of Cs, Os, and Is in the 13 letters, so the number is $\frac{13!}{2!2!2!} = \frac{13!}{8}$.

3. Six elements in the set means there are $2^6 = 64$ total subsets. However, this includes the empty subset, so there are 63 total nonempty subsets.

4. There are 12 total marbles in the bag, so the probability is $\frac{3}{12} = \frac{1}{4}$.

5. There are $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$ ways to select four of those seven students.

6. This probability would be $\frac{\binom{10}{3}}{\binom{15}{3}} = \frac{\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}}{\frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1}} = \frac{5 \cdot 3 \cdot 8}{5 \cdot 7 \cdot 13} = \frac{24}{91}$.

7. We imagine the worst possible selection. We could draw all 7 violets, 14 scarlets, 14 emeralds, and 14 ceruleans, which is $7 + 3 \cdot 14 = 49$ marbles without obtaining 15 of one color. By the pigeonhole principle, the next marble we will drawn must match one of the colors. Hence, we need 50.

8. Since 70% of 70 is 49, there are 49 people who indicate they get news from both television and internet. By inclusion-exclusion, the number who get news from either one is $60 + 70 - 49 = 81$. Thus, the number who do not get news from either source is $100 - 81 = 19$.

9. Any triangle in the figure is determined by three vertices, one of which must be A. The other two vertices can be chosen in $\binom{n}{2} = \frac{n(n-1)}{2}$ ways. Since there are 136 triangles, we have $n(n-1) = 272 = 17 \cdot 16 \Rightarrow n = 17$ (or $n = -16$, though this answer does not make sense in the context of this problem).

10. There are 11 flowers total, so there are $\frac{11!}{4!4!3!} = \frac{3 \cdot 11 \cdot 10 \cdot \cancel{9} \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}!}{\cancel{4}! \cdot \cancel{4}! \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 11550$

indistinguishable ways to line up the flowers.

11. Forcing roses on both ends of the row means arranging the remaining 9 flowers, which can

be done in $\frac{9!}{1!4!4!} = \frac{2 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}!}{\cancel{1} \cdot \cancel{4}! \cdot \cancel{4}! \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 630$ indistinguishable ways to line up the flowers with

roses on both ends.

12. There are $\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$ total ways to choose the team. Of those, there is 1 way to

choose all four juniors, and there are $\binom{5}{4} = 5$ ways to choose four seniors. Therefore, the

probability there is at least one junior and at least one senior is $\frac{126 - 1 - 5}{126} = \frac{120}{126} = \frac{20}{21}$.

13. Every term is of the form $kw^ax^by^cz^d$, where k, a, b, c , and d are nonnegative integers with $a + b + c + d = 10$. By the distribution method of sticks and stones, the number of solutions to

that equation is $\binom{4 + 10 - 1}{10} = \binom{13}{10} = \binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = 286$, so this is also the number of

terms in the expansion.

14. The probability of flipping a heads is $\frac{1}{2}$, so you expect it to take you 2 flips to get a heads.

15. The total number of ways to assign the servers is $\binom{13}{8} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1287$. The

number of ways to assign Jayden and Riley together is $\binom{11}{6} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462$ (since we

must choose ix from the remaining 11 servers). Therefore, the answer is $1287 - 462 = 825$

16. We treat the lilacs as one “super-lilac”. Thus, we are essentially arranging 1 lilac with 4

goldenrods and 3 roses. This can be done in $\frac{8!}{1!4!3!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}!}{\cancel{1} \cdot \cancel{4}! \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 280$ distinct ways.

17. Each ant has two choices of direction, so the number of possible outcomes is $2^7 = 128$. However, the only favorable outcomes are those where the ants all decide to travel in the same direction. Thus, there are 2 favorable outcomes, making the probability $\frac{2}{128} = \frac{1}{64}$.

18. From problem 10, there are 11550 possible arrangements of the eleven flowers. To ensure no two goldenrods are next to each other, we first arrange the roses and lilacs only; this can be done in $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$ ways. Now we have eight spots to place goldenrods: between the

flower or on either end of the row. Placing the goldenrods can be done in $\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$

ways. Thus, the number of ways to have no two goldenrods next to each other is $35 \cdot 70$

$= 2450$ ways. This makes the probability $\frac{2450}{11550} = \frac{245}{1155} = \frac{49}{231} = \frac{7}{33}$.

19. There are two cases depending on whether the last digit is 0 or not. If the number ends with a 0, then there are 9 choices for the first digit, 8 choices for the second, and 7 for the third, which can be done in $9 \cdot 8 \cdot 7 \cdot 1 = 504$ ways. If the number is even ending with something other than 0, then there are 4 choices for the last digit (2, 4, 6, or 8), 8 choices for the first digit (not 0 or the last digit), 8 for the second digit, and 7 for the third digit, which can be done in $8 \cdot 8 \cdot 7 \cdot 4 = 1792$ ways. Hence there are $504 + 1792 = 2296$ such numbers.

20. Letting p be the probability that a 2 is flipped, $1 - p$ is the probability that a 3 is flipped.

Therefore, $2^2 p + 3^2 (1 - p) = 7 \Rightarrow -5p = -2 \Rightarrow p = \frac{2}{5}$. Hence, $E(X^4) = 2^4 \left(\frac{2}{5}\right) + 3^4 \left(\frac{3}{5}\right) = \frac{32 + 243}{5} = \frac{275}{5} = 55$.

21. Let the subset with the desired property be T . Take any 677 numbers from S to form T .

Now consider the following partition of S into 676 subsets: $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, ...,

$\{2023, 2024, 2025\}$, $\{2026\}$. By the pigeonhole principle, at least 2 of these 677 numbers will come from one of the same three-element subsets listed above. Say these two numbers are m and n . If $m - n = 2$, then the sum must be even since the difference is, and thus the difference divides the sum. Hence, we cannot have 677 numbers in T . Indeed, if

$T = \{3k + 1 \mid k = 0, 1, 2, \dots, 675\}$, then T has the property claimed, and it has 676 elements. It follows that the largest number of elements possible in T is 676.

22. Let s be the number of seniors amongst this group, then $12 - s$ is the number of juniors amongst this group. Since the group is either 1 senior and 2 juniors or 2 seniors and 1 junior, we must have $160 = \binom{s}{2} \binom{12-s}{1} + \binom{s}{1} \binom{12-s}{2} = \frac{s(s-1)(12-s)}{2} + \frac{s(12-s)(11-s)}{2} = 5s(12-s)$
 $\Rightarrow 32 = s(12-s)$, and the only options for s are 4 or 8, making the number of juniors 8 or 4, respectively. The difference, then, is 4.

23. We calculate the probability that no stacks will have the same number, and subtract from 1. Of the possible ways to select the cards, which can be done in $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 36$ ways, there are only 12 ways to get stacks with different cards. This is because the first card from bag one can be any of the cards, the first cards from bag two can be either of the two cards that do not match the first card, then the second card from bag one can be either one left. But the next card from bag two is now determined—it cannot be the one that will match card two from bag one, but we cannot leave in the second bag the last card that will match the last card from bag one. Hence, we only have $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 12$ way to make the piles different. The probability, therefore, is $1 - \frac{12}{36} = 1 - \frac{1}{3} = \frac{2}{3}$.

24. The number of ways the darts can land is $\binom{16}{4} = \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = 1820$. The number of favorable outcomes is 10, since there are 4 horizontal, 4 vertical, and 2 diagonal lines. Therefore, the probability is $\frac{10}{1820} = \frac{1}{182}$.

25. Either Aiden and Billie are both telling the truth or both lying. In the first instance, the probability is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$; in the second instance, the probability is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$. Therefore, the

probability he was actually telling the truth is $\frac{\frac{1}{16}}{\frac{1}{16} + \frac{9}{16}} = \frac{1}{10}$.

26. There are two exclusive cases: you chose the right answer at the beginning, or you didn't. If you chose the correct answer with probability 80%, then you still get it right if the two choices that will be swapped do not include the correct answer. Thus, there must be two choices out of the four incorrect choices to swap— $\binom{4}{2} = 6$ ways—instead of two of the five total choices—

$\binom{5}{2} = 10$ ways. Hence this happens with probability $\frac{6}{10} = \frac{3}{5}$. It follows that you get the question correct by choosing the right answer with probability $80\% \cdot \frac{3}{5} = \frac{4}{5} \cdot \frac{3}{5} = \frac{12}{25}$. If you chose an incorrect answer with probability 20%, then to get the question correct, the swap must include your wrong answer and the correct answer. This happens with probability $\frac{1}{10}$. It follows that you get the question correct by choosing the wrong answer with probability $20\% \cdot \frac{1}{10} = \frac{1}{5} \cdot \frac{1}{10} = \frac{1}{50}$. Therefore, the probability that you get the question correct is $\frac{12}{25} + \frac{1}{50} = \frac{1}{2}$.

27. There are two cases that makes all the variables positive integers: $x_1 + x_2 + x_3 = 7$ and $y_1 + y_2 = 13$, or $x_1 + x_2 + x_3 = 13$ and $y_1 + y_2 = 7$. To determine the number of positive integer solutions, we encode the problem as a distribution (sticks and stones) problem. For instance in the first case, assign one stone to each of the three categories (variables), then determine the number of ways to distribute the remaining 4 stones, which is $\binom{3+4-1}{4} = \binom{6}{4} = \frac{6 \cdot 5}{2 \cdot 1} = 15$.

Similarly, for the second factor, there are $\binom{2+11-1}{11} = \binom{12}{11} = 12$ ways to distribute the stones, so the total in this scenario is $15 \cdot 12 = 180$. In the second case, assign one stone to each of the three categories (variables), then determine the number of ways to distribute the remaining 10 stones, which is $\binom{3+10-1}{10} = \binom{12}{10} = \frac{12 \cdot 11}{2 \cdot 1} = 66$; for the second factor, there are

$\binom{2+5-1}{5} = \binom{6}{5} = 6$ ways to distribute the stones, so the total in this scenario is $66 \cdot 6 = 396$.

Therefore, there are a total of $180 + 396 = 576$ such quintuplets.

28. Let the segment be the interval $[0, 2]$, and let $t > 1$ be the length of the longer piece. Therefore, $2 - t$ would be the length of the shorter piece, and since t is uniformly distributed on $[1, 2]$, the expected value of the ratio of the lengths would be $\int_1^2 \frac{2-t}{t} dt = \int_1^2 \left(\frac{2}{t} - 1 \right) dt$
 $= (2\ln|t| - t) \Big|_1^2 = (2\ln 2 - 2) - (-1) = 2\ln 2 - 1$.

29. Suppose we simply start choosing colors. The first sector can be colored in 11 ways, the second in 10 ways, the third in 10 ways, and so on. There are apparently $11 \cdot 10^6$ ways. However, the first and seventh sector could be the same color. Let C_n be the number of ways to color n sectors. If the first and last have different colors, we have C_7 . If they have the same colors, we can fuse the two sectors together, and we have a requested coloring of six sectors, C_6 . We have found that $C_7 + C_6 = 11 \cdot 10^6$. By similar reasoning, we recursively have $C_6 + C_5 = 11 \cdot 10^5$, $C_5 + C_4 = 11 \cdot 10^4$, $C_4 + C_3 = 11 \cdot 10^3$, $C_3 + C_2 = 11 \cdot 10^2$, and $C_2 = 11 \cdot 10$. By alternately adding and subtracting, we may compute C_7 : $C_7 = (C_7 + C_6) - (C_6 + C_5) + (C_5 + C_4) - (C_4 + C_3) + (C_3 + C_2) - C_2 = 11 \cdot 10^6 - 11 \cdot 10^5 + 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 11 \cdot 10$

$$= 11 \left(\frac{10^7 - 10}{10 + 1} \right) = 10^7 - 10 = 9,999,990.$$

30. Let each problem on Relay k be p_k , for $k = 1, 2, \dots, 2026$, so that there are six p_k 's on each test. Then the problem becomes the number of ways to arrange a string of six p_1 's, six p_2 's, and so on, up to six p_{2026} 's in a line. Now, we see that when problems from a particular test were solved, there is a uniquely determined way that the problems can be answered disregarding the solver. That is, wherever the six p_1 's are in the list, the first one appearing in the list must have been the first one solved. Disregarding the solver, there are

$$\binom{12156}{6} \binom{12150}{6} \binom{12144}{6} \dots \binom{6}{6} = \frac{12156!}{6!^{2026}} \text{ ways to create such an arrangement. Let's consider}$$

the ways the six students can solve the problems. Any of the six solvers can solve any of the 12156 problems, so they can solve the problems in 6^{12156} ways. Thus we have

$$N = \frac{12156! \cdot 6^{12156}}{6!^{2026}} = \frac{12156! \cdot 6^{10130}}{5^{2026} 4^{2026} 3^{2026} 2^{2026}} = \frac{12156! \cdot 3^{10130} 2^{10130}}{5^{2026} 2^{4052} 3^{2026} 2^{2026}} = \frac{12156! \cdot 3^{8104} 2^{4052}}{5^{2026}}.$$

Now, counting the factors of 3, there are

$$\left\lfloor \frac{12156}{3} \right\rfloor + \left\lfloor \frac{12156}{9} \right\rfloor + \left\lfloor \frac{12156}{27} \right\rfloor + \left\lfloor \frac{12156}{81} \right\rfloor + \left\lfloor \frac{12156}{243} \right\rfloor + \left\lfloor \frac{12156}{729} \right\rfloor$$

$$+ \left\lfloor \frac{12156}{2187} \right\rfloor + \left\lfloor \frac{12156}{6561} \right\rfloor = 4052 + 1350 + 450 + 150 + 50 + 16 + 5 + 1 = 6074 \text{ factors of 3 in } 12156!.$$

Therefore, there are a total of $6074 + 8104 = 14178$ factors of 3 in N . The sum of the digits in this number is $1 + 4 + 1 + 7 + 8 = 21$.