

Answers:

1. D
2. E
3. A
4. W
5. P
6. O
7. R
8. L
9. A
10. W
11. E
12. F
13. M
14. B
15. Y
16. X
17. A
18. S
19. G
20. F

Solutions:

For all problems, a reminder that α and β are first quadrant angles.

$$1. \frac{\sqrt{\csc^2 \alpha - 1}}{\csc \alpha} = \frac{\sqrt{\cot^2 \alpha}}{\csc \alpha} = \frac{\cot \alpha}{\csc \alpha} = \frac{\frac{\cos \alpha}{\sin \alpha}}{\frac{1}{\sin \alpha}} = \cos \alpha$$

$$2. \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1 - \tan^2 \alpha}{\sec^2 \alpha} = \frac{1}{\sec^2 \alpha} - \frac{\tan^2 \alpha}{\sec^2 \alpha} = \cos^2 \alpha - \frac{\frac{\sin^2 \alpha}{\cos^2 \alpha}}{\frac{1}{\cos^2 \alpha}} = \cos^2 \alpha - \sin^2 \alpha = \cos(2\alpha)$$

$$3. \sec^2 \alpha - \tan^2 \alpha = 1$$

$$4. \tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$5. \cos \alpha \cdot \cot \alpha = \cos \alpha \cdot \frac{\cos \alpha}{\sin \alpha} = \frac{\cos^2 \alpha}{\sin \alpha} = \frac{1 - \sin^2 \alpha}{\sin \alpha} = \frac{1}{\sin \alpha} - \frac{\sin^2 \alpha}{\sin \alpha} = \csc \alpha - \sin \alpha$$

$$\begin{aligned} 6. & (\cos \alpha + \sin \alpha)(\cot \alpha + \tan \alpha) = (\cos \alpha + \sin \alpha) \left(\frac{\cos \alpha}{\sin \alpha} + \frac{\sin \alpha}{\cos \alpha} \right) \\ & = (\cos \alpha + \sin \alpha) \left(\frac{\cos^2 \alpha + \sin^2 \alpha}{\sin \alpha \cdot \cos \alpha} \right) = (\cos \alpha + \sin \alpha) \left(\frac{1}{\sin \alpha \cdot \cos \alpha} \right) = \frac{\cos \alpha}{\sin \alpha \cdot \cos \alpha} + \frac{\sin \alpha}{\sin \alpha \cdot \cos \alpha} \\ & = \frac{1}{\sin \alpha} + \frac{1}{\cos \alpha} = \csc \alpha + \sec \alpha \end{aligned}$$

$$\begin{aligned} 7. & \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta) = \frac{1}{2} (\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta) + \frac{1}{2} (\cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta) \\ & = \frac{1}{2} (\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta + \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta) = \frac{1}{2} (\cos \alpha \cdot \cos \beta + \cos \alpha \cdot \cos \beta) \\ & = \frac{1}{2} (2 \cos \alpha \cdot \cos \beta) = \cos \alpha \cdot \cos \beta \end{aligned}$$

$$\begin{aligned} 8. & 4 \sin \alpha \cdot \cos \alpha - 8 \sin^3 \alpha \cdot \cos \alpha = 4 \sin \alpha \cdot \cos \alpha (1 - 2 \sin^2 \alpha) = 2 \cdot 2 \sin \alpha \cdot \cos \alpha (1 - 2 \sin^2 \alpha) \\ & 2 \cdot 2 \sin \alpha \cdot \cos \alpha (1 - 2 \sin^2 \alpha) = 2 \sin(2\alpha) \cdot \cos(2\alpha) = \sin(2 \cdot 2\alpha) = \sin(4\alpha) \end{aligned}$$

$$9. \sin^2 \alpha + \cos^2 \alpha = 1$$

$$10. \frac{1 - \cos \alpha}{\sin \alpha} = \tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\begin{aligned} 11. \frac{\sin(3\alpha)}{2\sin \alpha} - \frac{1}{2} &= \frac{\sin(2\alpha + \alpha)}{2\sin \alpha} - \frac{1}{2} = \frac{\sin(2\alpha) \cdot \cos \alpha + \cos(2\alpha) \cdot \sin \alpha}{2\sin \alpha} - \frac{1}{2} \\ &= \frac{2\sin \alpha \cdot \cos \alpha \cdot \cos \alpha + (1 - 2\sin^2 \alpha) \cdot \sin \alpha}{2\sin \alpha} - \frac{1}{2} = \frac{2\sin \alpha \cdot \cos^2 \alpha + \sin \alpha - 2\sin^3 \alpha}{2\sin \alpha} - \frac{1}{2} \\ &= \frac{2\sin \alpha (1 - \sin^2 \alpha) + \sin \alpha - 2\sin^3 \alpha}{2\sin \alpha} - \frac{1}{2} = \frac{2\sin \alpha - 2\sin^3 \alpha + \sin \alpha - 2\sin^3 \alpha}{2\sin \alpha} - \frac{1}{2} \\ &= \frac{3\sin \alpha - 4\sin^3 \alpha}{2\sin \alpha} - \frac{1}{2} = \frac{3\sin \alpha}{2\sin \alpha} - \frac{4\sin^3 \alpha}{2\sin \alpha} - \frac{1}{2} = \frac{3}{2} - 2\sin^2 \alpha - \frac{1}{2} = 1 - 2\sin^2 \alpha = \cos(2\alpha) \end{aligned}$$

$$\begin{aligned} 12. 4\cos^3 \alpha - 3\cos \alpha &= 2\cos^3 \alpha - \cos \alpha - 2\cos \alpha + 2\cos^3 \alpha = (2\cos^2 \alpha - 1)\cos \alpha \\ &= (1 - \cos^2 \alpha)2\cos \alpha = \cos(2\alpha)\cos \alpha - \sin^2 \alpha \cdot 2\cos \alpha = \cos(2\alpha)\cos \alpha - 2\sin \alpha \cdot \cos \alpha \cdot \sin \alpha \\ &= \cos(2\alpha)\cos \alpha - \sin(2\alpha)\sin \alpha = \cos(2\alpha + \alpha) = \cos(3\alpha) \end{aligned}$$

$$\begin{aligned} 13. (\csc \alpha - \cot \alpha)(\sec \alpha + 1) &= \left(\frac{1}{\sin \alpha} - \frac{\cos \alpha}{\sin \alpha} \right) \left(\frac{1}{\cos \alpha} + \frac{\cos \alpha}{\cos \alpha} \right) = \left(\frac{1 - \cos \alpha}{\sin \alpha} \right) \left(\frac{1 + \cos \alpha}{\cos \alpha} \right) \\ &= \frac{1 - \cos^2 \alpha}{\sin \alpha \cdot \cos \alpha} = \frac{\sin^2 \alpha}{\sin \alpha \cdot \cos \alpha} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha \end{aligned}$$

$$\begin{aligned} 14. \csc^2 \alpha \cdot \sec^2 \alpha - \tan^2 \alpha - \cot^2 \alpha &= (1 + \cot^2 \alpha)(1 + \tan^2 \alpha) - \tan^2 \alpha - \cot^2 \alpha = 1 + \cot^2 \alpha \\ &+ \tan^2 \alpha + \tan^2 \alpha \cdot \cot^2 \alpha - \tan^2 \alpha - \cot^2 \alpha = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} 15. \text{Using a result from \#12, } 1 + 3\cos \alpha + \cos(2\alpha) + \cos(3\alpha) &= 1 + 3\cos \alpha + 2\cos^2 \alpha - 1 \\ &+ 4\cos^3 \alpha - 3\cos \alpha = 2\cos^2 \alpha + 4\cos^3 \alpha \end{aligned}$$

$$16. \frac{\sin \alpha}{\sin \alpha + \cos \alpha} = \frac{\frac{\sin \alpha}{\cos \alpha}}{\frac{\sin \alpha + \cos \alpha}{\cos \alpha}} = \frac{\tan \alpha}{\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\cos \alpha}} = \frac{\tan \alpha}{\tan \alpha + 1} = \frac{\tan \alpha}{1 + \tan \alpha}$$

$$\begin{aligned} 17. 2\sin^2 \alpha - \sin^4 \alpha + \cos^4 \alpha &= 2\sin^2 \alpha + \cos^4 \alpha - \sin^4 \alpha = 2\sin^2 \alpha \\ &+ (\cos^2 \alpha - \sin^2 \alpha)(\cos^2 \alpha + \sin^2 \alpha) = 2\sin^2 \alpha + (\cos^2 \alpha - \sin^2 \alpha) \cdot 1 = 2\sin^2 \alpha + \cos^2 \alpha - \sin^2 \alpha \end{aligned}$$

$$= \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\begin{aligned} 18. \quad & 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cdot \cos\left(\frac{\alpha - \beta}{2}\right) = 2 \cos\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) \cdot \cos\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) = 2 \left(\cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} - \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \right) \\ & \cdot \left(\cos\frac{\alpha}{2} \cdot \cos\frac{\beta}{2} + \sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \right) = 2 \left(\cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2} - \sin^2\frac{\alpha}{2} \cdot \sin^2\frac{\beta}{2} \right) \\ & = 2 \left(\cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2} - \left(1 - \cos^2\frac{\alpha}{2}\right) \left(1 - \cos^2\frac{\beta}{2}\right) \right) \\ & = 2 \left(\cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2} - \left(1 - \cos^2\frac{\alpha}{2} - \cos^2\frac{\beta}{2} + \cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2}\right) \right) \\ & = 2 \left(\cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2} - 1 + \cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} - \cos^2\frac{\alpha}{2} \cdot \cos^2\frac{\beta}{2} \right) = 2 \left(-1 + \cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} \right) \\ & = 2 \left(-1 + \frac{1 + \cos \alpha}{2} + \frac{1 + \cos \beta}{2} \right) = -2 + 1 + \cos \alpha + 1 + \cos \beta = \cos \alpha + \cos \beta \end{aligned}$$

$$\begin{aligned} 19. \quad & 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1 = 8 \cos^2 \alpha (\cos^2 \alpha - 1) + 1 = 8 \cos^2 \alpha (-\sin^2 \alpha) + 1 = 1 - 8 \sin^2 \alpha \cdot \cos^2 \alpha \\ & = 1 - 2 \cdot 4 \sin^2 \alpha \cdot \cos^2 \alpha = 1 - 2 (2 \sin \alpha \cdot \cos \alpha)^2 = 1 - 2 \sin^2 (2\alpha) = \cos(2 \cdot 2\alpha) = \cos(4\alpha) \end{aligned}$$

$$\begin{aligned} 20. \quad & 4 \cos \alpha \cdot \cos\left(\frac{\pi}{3} - \alpha\right) \cdot \cos\left(\frac{\pi}{3} + \alpha\right) = 4 \cos \alpha \cdot \left(\cos\frac{\pi}{3} \cdot \cos \alpha + \sin\frac{\pi}{3} \cdot \sin \alpha \right) \\ & \cdot \left(\cos\frac{\pi}{3} \cdot \cos \alpha - \sin\frac{\pi}{3} \cdot \sin \alpha \right) = 4 \cos \alpha \left(\cos^2\frac{\pi}{3} \cdot \cos^2 \alpha - \sin^2\frac{\pi}{3} \cdot \sin^2 \alpha \right) \\ & = 4 \cos \alpha \left(\frac{1}{4} \cos^2 \alpha - \frac{3}{4} (1 - \cos^2 \alpha) \right) = 4 \cos \alpha \left(\frac{1}{4} \cos^2 \alpha - \frac{3}{4} + \frac{3}{4} \cos^2 \alpha \right) \\ & = 4 \cos \alpha \left(\cos^2 \alpha - \frac{3}{4} \right) = 4 \cos^3 \alpha - 3 \cos \alpha, \text{ which is equivalent to \#12, so this equals } \cos(3\alpha). \end{aligned}$$