

1. 45360
2. $\frac{5}{9}$
3. 34
4. 1
5. 41
6. x
7. $\frac{3}{4}$ or 0.75
8. $\frac{13}{3}$
9. 33
10. 6:1
11. 2036
12. $\sqrt{3}$
13. 6
14. 12
15. 165
16. $\frac{1}{24}$
17. 99
18. $4\sqrt{33}$
19. 3
20. 7
21. 7
22. $(-\infty, 8]$
23. 24π
24. 19
25. $9\frac{1}{2}$ or 45.5

$$1. \frac{9!}{2!2!2!} = \frac{362880}{8} = 45360$$

$$2. \quad ab + a + b \equiv 2 \pmod{3}$$

$$\Rightarrow (a+1)(b+1) = ab + a + b + 1$$

$$\equiv 0 \pmod{3}, \text{ so } a \text{ or } b \text{ (or both) are}$$

$$\text{congruent to } 2 \pmod{3}. \text{ Neither}$$

$$\text{are congruent to } 2 \pmod{3} \text{ has}$$

$$\text{probability } \left(\frac{2}{3}\right)^2 = \frac{4}{9}, \text{ so the}$$

$$\text{complement has probability}$$

$$1 - \frac{4}{9} = \frac{5}{9}.$$

$$3. \quad \binom{8}{4} = 70 \text{ total paths from } (0,0) \text{ to}$$

$$(4,4); \binom{4}{2} = 6 \text{ paths from } (0,0) \text{ to}$$

$$(2,2); \binom{4}{2} = 6 \text{ paths from } (2,2) \text{ to}$$

$$(4,4); \text{ complement counting gives}$$

$$70 - 6 \cdot 6 = 34 \text{ such paths}$$

$$4. \quad 9^2 + 4 \cdot 12^2 + 4^5 \equiv 1 + 4 \cdot 4 + 4$$

$$\equiv 1 \pmod{10}, \text{ so the units digit is } 1$$

$$5. \quad 9^2 + 4 \cdot 12^2 + 4^5 = (9 + 2 \cdot 4^2)^2$$

$$= 41^2, \text{ so the greatest prime factor}$$

$$\text{is } 41$$

$$6. \quad f^1(x) = \frac{1}{1-x}; \quad f^2(x) = \frac{x-1}{x};$$

$$f^3(x) = x, \text{ so these functions cycle}$$

$$\text{every 3 compositions. Since } 2025$$

$$\text{is divisible by } 3, f^{2025}(x) = x.$$

$$7. \quad \text{If she flips a head first, she will}$$

$$\text{definitely flip HT before TT, so that}$$

$$\text{has probability } \frac{1}{2}. \text{ If she flips a tail}$$

$$\text{first, she must immediately flip a}$$

$$\text{head next in order to flip HT}$$

$$\text{before TT, so this has probability}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}. \text{ Therefore, the}$$

$$\text{probability is } \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

$$8. \quad a = \log x, b = \log y \Rightarrow 64 = 4^3$$

$$= (a+b)^3 = a^3 + b^3 + 3ab(a+b) = 12 + 12ab$$

$$\Rightarrow ab = \frac{13}{3}$$

$$9. \quad 25 \text{ multiples of } 4, 16 \text{ multiples of } 6,$$

$$\text{and } 8 \text{ multiples of both (multiples}$$

$$\text{of } 12), \text{ so there are } 25 + 16 - 8 = 33$$

$$\text{total such integers}$$

10. Using mass point geometry, assign masses of 3 to points B and C and mass 1 to point A . Therefore, the mass at point D would be 6, and $AP:PD = 6:1$.

$$11. \sum_{k=0}^9 ((10-k) \cdot 2^k) = 10 + 18 + 32 + 56 + 9 + 160 + 256 + 384 + 512 + 512 = 2036$$

12. The solutions are the imaginary sixth roots of unity, so the longest side is one of the shorter diagonals of a regular hexagon with side length 1, so this is $\sqrt{3}$.

$$13. x = 1 + \frac{1}{2 + \frac{1}{x}} \Rightarrow x = 1 + \frac{x}{2x+1}$$

$$2x^2 - 2x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{3}}{2}, \text{ but}$$

since x is formed only by addition,

$$x = \frac{1 + \sqrt{3}}{2} \Rightarrow a + b + c = 1 + 3 + 2 = 6$$

14. Using the AM-GM inequality,

$$\frac{1}{3} \left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \geq \sqrt[3]{\frac{a}{bc} \cdot \frac{b}{ac} \cdot \frac{c}{ab}}$$

$$= \sqrt[3]{\frac{1}{abc}} \geq \sqrt[3]{4^3} = 4 \Rightarrow \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}$$

≥ 12 , with equality occurring

$$\text{when } a = b = c = \frac{1}{4}.$$

15. For four digit number $abcd$,

$$a + b + c = d, \text{ and } 1 \leq a \leq 9 \text{ while}$$

$$0 \leq b, c \leq 8. \text{ Setting } a' = a - 1, \text{ we}$$

$$\text{have } a' + b + c = d - 1, \text{ where}$$

$$0 \leq a', b, c \leq 8. \text{ By Stars and Bars,}$$

the number of ways to achieve this

$$\text{result is } \binom{3 + (d-1) - 1}{3-1} = \binom{d+1}{2},$$

$$\text{so } \sum_{d=1}^9 \binom{d+1}{2} = 1 + 3 + 6 + 10 + 15$$

$$+ 21 + 28 + 36 + 45 = 165.$$

16. Let s be a side length of the

hexagon, then the area enclosed by

$$\text{the triangle is } \frac{1}{2} \left(\frac{s}{2} \right)^2 \sin 120^\circ$$

$$= \frac{s^2 \sqrt{3}}{16}. \text{ The area enclosed by the}$$

$$\text{hexagon is } \frac{3s^2 \sqrt{3}}{2}, \text{ so the fraction}$$

$$\text{is } \frac{\frac{s^2\sqrt{3}}{3\cancel{s^2\sqrt{3}}}}{\frac{2}{2}} = \frac{1}{16} \cdot \frac{2}{3} = \frac{1}{24}.$$

17. Because 19 and 100 are relatively prime, $19^{40} = 19^{\Phi(100)} \equiv 1 \pmod{100}$
 $\Rightarrow 19^{120} \equiv 1 \pmod{100} \Rightarrow 19^{125}$
 $\equiv 19^5 \pmod{100}$, so just figure out the last two digits of 19^5 . 19^2 ends in 61, so 19^4 ends in the same two digits as 61^2 , which ends in 21. Therefore, 19^5 ends in the same two digits as $21 \cdot 19$, which ends in 99.

$$18. \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 2 \\ 0 & 5 & 2 \end{vmatrix} = \langle -8, -8, 20 \rangle, \text{ so}$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{(-8)^2 + (-8)^2 + 20^2} \\ = \sqrt{528} = 4\sqrt{33}$$

19. After t minutes, Isabella has traveled $\frac{t}{20}$ miles. Melissa starts at $t=10$ minutes and runs each mile in 10 minutes, then rests 5 minutes, so she completes each mile in 15 minutes; at time t , then, Melissa has run $\left\lfloor \frac{t-10}{15} \right\rfloor$ for

$$t \geq 10. \text{ Therefore, } \frac{t}{20} = \left\lfloor \frac{t-10}{15} \right\rfloor$$

$$\Rightarrow \frac{t}{20} \leq \frac{t-10}{15} < \frac{t}{20} + 1 \Rightarrow 3t \leq 4t$$

$$-40 < 3t + 60 \Rightarrow 40 \leq t < 100, \text{ but}$$

the only values in this interval that

make $\frac{t}{20}$ an integer are 40, 60,

and 80, so they meet 3 times.

20. The expected value of the sum is

$$(2+12) \cdot \frac{1}{36} + (3+11) \cdot \frac{2}{36} \\ + (4+10) \cdot \frac{3}{36} + (5+9) \cdot \frac{4}{36} \\ + (6+8) \cdot \frac{5}{36} + (7) \cdot \frac{6}{36} = 7.$$

21. $\frac{x^3+8}{x^2-4} = x + \frac{4}{x-2}$, so the largest x -value that keeps this an integer is $x=6$ --anything greater than that makes $\frac{4}{x-2}$ a fraction. Plugging this value in yields a value of 7.

22. $y = -4\left(x - \frac{3}{2}\right)^2 + 8$ is a parabola opening downward with vertex with y -coordinate 8. Therefore, the range is $(-\infty, 8]$.

23. The standard form of this ellipse is

$$\frac{x^2}{16} + \frac{y^2}{36} = 1, \text{ so the enclosed area is}$$

$$\pi \cdot \sqrt{16} \cdot \sqrt{36} = 24\pi.$$

24. Since 5 and 6 are relatively prime,
the largest number that cannot be
found as a linear combination of 5
and 6 is $5 \cdot 6 - 5 - 6 = 19$.

$$25. \sum_{x=1}^{90} \sin^2 x = \sum_{x=1}^{44} \sin^2 x + \frac{1}{2} + \sum_{x=46}^{89} \sin^2 x$$

$$+1 = \sum_{x=1}^{44} \sin^2 x + \frac{1}{2} + \sum_{x=1}^{44} \cos^2 x + 1$$

$$= 44 + \frac{1}{2} + 1 = \frac{91}{2}$$