

Answers:

1. B
2. A
3. C
4. E
5. B
6. A
7. C
8. B
9. E
10. B
11. D
12. E
13. D
14. A
15. B
16. A
17. B
18. D
19. E
20. C
21. D
22. D
23. A
24. C
25. C
26. A
27. B
28. E
29. B
30. D

Solutions:

$$1. \frac{2026(2026^{2026})^{2026}}{(2026 \cdot 2026^{2026})^{2026}} = \frac{2026 \cancel{(2026^{2026})^{2026}}}{2026^{2026} \cancel{(2026^{2026})^{2026}}} = 2026^{1-2026} = 2026^{-2025} \Rightarrow k = -2025$$

$$2. \log_{x+1} \sqrt{11} = \log_{x+1} y = \log_{x+1} k \cdot \log_k y = -1 \cdot \frac{1}{4} = -\frac{1}{4} \Rightarrow (x+1)^{-\frac{1}{4}} = \sqrt{11} \Rightarrow x+1 = (\sqrt{11})^{-4} = \frac{1}{121}$$

$$\Rightarrow x = -\frac{120}{121}$$

$$3. 0 = 2^{2x+1} - 2049 \cdot 2^x + 2^{10} = 2 \cdot (2^x)^2 - 2049 \cdot 2^x + 1024 = (2 \cdot 2^x - 1)(2^x - 1024) \Rightarrow 2^x = \frac{1}{2} \text{ or } 1024$$

$$\Rightarrow x = -1 \text{ or } 10, \text{ so the product of the solutions is } -10.$$

$$4. 1 = \log_5(x+1) - \log_{25} x = \log_{25}(x+1)^2 - \log_{25} x = \log_{25} \frac{(x+1)^2}{x} \Rightarrow \frac{(x+1)^2}{x} = 25^1 = 25$$

$$x^2 + 2x + 1 = (x+1)^2 = 25x \Rightarrow x^2 - 23x + 1 = 0 \Rightarrow x = \frac{23 \pm \sqrt{(-23)^2 - 4}}{2} = \frac{23 \pm \sqrt{525}}{2} = \frac{23 \pm 5\sqrt{21}}{2}.$$

Since both of these solutions keep the arguments of the logarithms positive, they both work.

$$5. 20^{26} \cdot 26^{26} \cdot 20^{20} \cdot 26^{20} = 20^{26+20} \cdot 26^{26+20} = 20^{46} \cdot 26^{46} = (20 \cdot 26)^{46} = 520^{46}$$

$$6. \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \dots \cdot \log_{81} 80 = \frac{\ln 3}{\ln 4} \cdot \frac{\ln 4}{\ln 5} \cdot \frac{\ln 5}{\ln 6} \cdot \dots \cdot \frac{\ln 80}{\ln 81} = \frac{\ln 3}{\ln 81} = \log_{81} 3 = \frac{1}{4}$$

$$7. 2^{8000} = (2^8)^{1000} = 256^{1000}; 3^{7000} = (3^7)^{1000} = 2187^{1000}; 4^{6000} = (4^6)^{1000} = 4096^{1000}; 5^{5000} = (5^5)^{1000}$$

$$3125^{1000}; 6^{4000} = (6^4)^{1000} = 1296^{1000}; 7^{3000} = (7^3)^{1000} = 343^{1000}; 8^{2000} = (8^2)^{1000} = 64^{1000}. \text{ Since all of}$$

the quantities are now raised to the same power, we may simply arrange their bases:

$$64 < 256 < 343 < 1296 < 2187 < 3125 < 4096, \text{ so the middle term corresponds to } 1296, \text{ which is } 6^{4000}.$$

$$8. \log \frac{16}{225} = \log 16 - \log 225 = \log 2^4 - \log \frac{900}{4} = 4 \log 2 - (\log 900 - \log 4) = 4 \log 2 - (\log 9 + \log 100$$

$$- \log 2^2) = 4 \log 2 - \log 3^2 - 2 + 2 \log 2 = 6 \log 2 - 2 \log 3 - 2 \approx 6 \cdot 0.301 - 2 \cdot 0.477 - 2 = 1.806 - 0.954 - 2$$

$$= -1.148.$$

$$9. 3^{3^x} = 27^{81^9} = (3^3)^{(3^4)^9} = (3^3)^{3^{4 \cdot 9}} = 3^{3 \cdot 3^{36}} = 3^{3^{1+36}} = 3^{3^{37}} \Rightarrow x = 37$$

10. $0 = x^4 - 8x^3 + 25x^2 - 36x + 18 = (x-1)(x-3)(x^2 - 4x + 6)$, and the solutions to the quadratic are imaginary, so the sum of the real solutions is $1+3=4$, so $\log_2 4 = 2$.

11. Clearly, 1 is one of the solutions, so the other solution is also the product of the solutions.

$$x^{x^{\frac{3}{2}}} = x^{x\sqrt{x}} = (x\sqrt{x})^x = \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x} \Rightarrow x^{\frac{3}{2}} = \frac{3}{2}x \Rightarrow x^{\frac{1}{2}} = \frac{3}{2} \Rightarrow x = \frac{9}{4}, \text{ so this is the product of the solutions.}$$

12. Two of the elements ($\log_5 24$ and $\log_7 37$) are less than 2. $\log_3 43$ is greater than 3, and $\log_2 21$ is greater than 4. $\log_9 97$ is between 2 and 3, so this must be the middle term.

$$13. \text{ Using change of base, the equation becomes } \frac{\log x}{\log 3} \cdot \frac{\log x}{\log 4} \cdot \frac{\log x}{\log 6} = \frac{\log x}{\log 3} \cdot \frac{\log x}{\log 4} + \frac{\log x}{\log 3} \cdot \frac{\log x}{\log 6} + \frac{\log x}{\log 4} \cdot \frac{\log x}{\log 6} \\ \Rightarrow \frac{\log x}{\log 3 \log 4 \log 6} = \frac{1}{\log 3 \log 4} + \frac{1}{\log 3 \log 6} + \frac{1}{\log 4 \log 6} \Rightarrow \log x = \log 6 + \log 4 + \log 3 \\ = \log(6 \cdot 4 \cdot 3) = \log 72, \text{ so } 72 \text{ is a solution to this equation. However, notice that } 1 \text{ is also a solution, so the sum of the solutions is } 73.$$

14. Base 8 logarithms of the integers from 1 to 7 will be greater than or equal to 0 and less than 1, so the first seven terms of the sum will each be 0. Base 8 logarithms of the integers from 8 to 63 will be greater than or equal to 1 and less than 2, so the next $63-8+1=56$ terms will each be 1. Base 8 logarithms of the integers from 64 to 511 will be greater than or equal to 2 and less than 3, so the next $511-64+1=448$ terms will each be 2. The sum of these terms thus far is $56 \cdot 1 + 448 \cdot 2 = 952$, so we still need $2026 - 952 = 1074$ more to be added. Therefore, $3(n-512+1) = 1074 \Rightarrow n-511 = 358 \Rightarrow n = 869$.

$$15. \frac{1}{5050} = \log \left(\frac{(x)^{\frac{1}{x}}}{(x)^{\frac{1}{x+1}}} \right) = \log(x)^{\frac{1}{x}} - \log(x)^{\frac{1}{x+1}} = \frac{1}{x} \log x - \frac{1}{x+1} \log x = \frac{1}{x(x+1)} \log x = \log(x)^{\frac{1}{x(x+1)}} \\ \Rightarrow 10^{\frac{1}{5050}} = (x)^{\frac{1}{x(x+1)}}. \text{ Since } 5050 = 50 \cdot 101 = \frac{100 \cdot 101}{2}, \text{ this becomes } (x)^{\frac{1}{x(x+1)}} = 10^{\frac{2}{100 \cdot 101}}$$

$$= 100^{\frac{1}{100 \cdot 101}} \Rightarrow x = 100.$$

$$16. 10^x = 192 = 4^3 \cdot 3 = (10^a)^3 (10^b) = 10^{3a} \cdot 10^b = 10^{3a+b} \Rightarrow x = 3a + b$$

$$17. \log_{2026}(\log_{2024}(\log_{2022}(\log_{2020} x))) \text{ is defined if } \log_{2024}(\log_{2022}(\log_{2020} x)) > 0 \\ \Rightarrow \log_{2022}(\log_{2020} x) > 1 \Rightarrow \log_{2020} x > 2022 \Rightarrow x > 2020^{2022}, \text{ so this is the value of } c.$$

$$18. \text{Plugging the second equation into the first, } a^{8a} = (8a)^a \Rightarrow (a^8)^a = 8^a a^a \Rightarrow (a^a)^7 = 8^a \\ \Rightarrow (a^7)^a = 8^a \Rightarrow a^7 = 8 \Rightarrow a = \sqrt[7]{8}.$$

$$19. 8^{\log_2 x} - 5 \cdot 4^{\log_2 x} = 2 \cdot 2^{\log_2 x} - 2^{\log_2 24} \Rightarrow x^{\log_2 8} - 5 \cdot x^{\log_2 4} = 2 \cdot x^{\log_2 2} - 24 \Rightarrow x^3 - 5x^2 = 2x - 24 \\ 0 = x^3 - 5x^2 - 2x + 24 = (x-3)(x-4)(x+2), \text{ but } x \text{ can only be positive since it is the argument} \\ \text{of a logarithm. Therefore, the sum of the real solutions is } 3 + 4 = 7.$$

$$20. \text{Using the approximation from question 8, } 2^{2026} = (10^{\log 2})^{2026} = 10^{2026 \log 2} \approx 10^{2026(0.301)} = 10^{609.826} \\ = 10^{0.826} \cdot 10^{609}, \text{ and since } 1 < 10^{0.826} < 10, \text{ this must be some digit followed by 609 digits, so this} \\ \text{is a 610-digit number.}$$

$$21. 0.9m_0 = m_0 e^{k(2)} \Rightarrow 0.9 = e^{2k} \Rightarrow \ln 0.9 = 2k \Rightarrow k = \frac{1}{2} \ln 0.9, \text{ so plugging this into the equation}$$

$$\text{and solving for 10\% of the mass, } 0.1m_0 = m_0 e^{\frac{1}{2} \ln 0.9(t)} \Rightarrow 0.1 = e^{\frac{1}{2} \ln 0.9(t)} \Rightarrow \ln 0.1 = \frac{1}{2} \ln 0.9(t)$$

$$\Rightarrow t = \frac{2 \ln 0.1}{\ln 0.9} = \frac{2(-1)}{\log 9 - \log 10} = \frac{-2}{\log 9 - 1} = \frac{2}{1 - \log 9}.$$

$$22. 47^{x \log_{23} 68} = 68^{2 \log_{13} 47} = 47^{2 \log_{13} 68} \Rightarrow x \log_{23} 68 = 2 \log_{13} 68 \Rightarrow x = \frac{2 \log_{13} 68}{\log_{23} 68} = \frac{2 \log_{68} 23}{\log_{68} 13} = 2 \log_{13} 23 \\ = \log_{13} 23^2 = \log_{13} 529 \Rightarrow b + n = 13 + 529 = 542$$

$$23. 3^{x+1} - 18(3^{x-1})^2 = 1 \Rightarrow 3 \cdot 3^x - 2 \cdot 3^{2x} = 1 \Rightarrow 0 = 2 \cdot (3^x)^2 - 3 \cdot 3^x + 1 = (2 \cdot 3^x - 1)(3^x - 1) \\ \Rightarrow 3^x = \frac{1}{2} \text{ or } 3^x = 1 \Rightarrow x = \log_3 \frac{1}{2} \text{ or } x = 0, \text{ so the sum of the solutions is } \log_3 \frac{1}{2}.$$

24. $-6 = (\log_x 4 - 4)(\log_2 x + 2) = 2 + 2\log_x 4 - 4\log_2 x - 8 \Rightarrow \log_x 4 = 2\log_2 x \Rightarrow 2\log_x 2 = 2\log_2 x$
 $\Rightarrow \log_x 2 = \log_2 x \Rightarrow 1 = (\log_2 x)^2 \Rightarrow \pm 1 = \log_2 x \Rightarrow x = 2 \text{ or } \frac{1}{2}$, so the product of the solutions is 1.

25. $n = 3^m + 3^{m-1} = 3^m + \frac{1}{3} \cdot 3^m = \frac{4}{3} \cdot 3^m$, so $m = 1 \Rightarrow n = 4$ are the smallest positive integral values of m and n , and increasing either value would increase n^m , so $n^m = 4^1 = 4$.

26. By the remainder theorem, the remainder equals the value of the expression at $x = -1$.
 Therefore, the remainder is $5(-1)^{2n+1} - 10(-1)^{2n} + 3(-1)^{2n-1} + 5 = -5 - 10 - 3 + 5 = -13$.

27. $\sum_{k=1}^{100} (\log(2^k)) = \sum_{k=1}^{100} (k \cdot \log 2) = \log 2 \cdot \sum_{k=1}^{100} k = \log 2 \cdot \frac{100 \cdot 101}{2} = 5050 \log 2$, so using the approximation from problem 8 again, $5050 \log 2 \approx 5050(.301) = 1520.05$, so B is closest.

28. If $k = 0$, there is only one solution. Now, assuming $k \neq 0$, multiplying both sides of the equation by 2^x yields $k \cdot (2^x)^2 + 1 = 3 \cdot 2^x \Rightarrow 0 = k \cdot (2^x)^2 - 3 \cdot 2^x + 1 \Rightarrow 2^x = \frac{3 \pm \sqrt{9 - 4k}}{2k}$, so there is one solution if $k = \frac{9}{4}$. Now, if the numerator of this express yields one positive and one negative value, then there will also be only one solution, so $k < 0$ also gives just one solution. Therefore, the solution is $(-\infty, 0] \cup \left\{ \frac{9}{4} \right\}$.

29. Note that $72 = 2^3 \cdot 3^2$ and $315 = 3^2 \cdot 5 \cdot 7$. Writing the expression with exponents, we have $2^{\frac{3}{4}} \cdot 3^{\frac{1}{2}} \cdot a^{\frac{1}{4}} \cdot 3^{\frac{2}{3}} \cdot 5^{\frac{1}{3}} \cdot 7^{\frac{1}{3}} \cdot b^{\frac{1}{3}} = 2^{\frac{3}{4}} \cdot 3^{\frac{7}{6}} \cdot 5^{\frac{1}{3}} \cdot 7^{\frac{1}{3}} \cdot a^{\frac{1}{4}} \cdot b^{\frac{1}{3}}$. For this to be an integer, we need $a^{\frac{1}{4}} \cdot b^{\frac{1}{3}} = 2^{\frac{1}{4}} \cdot 3^{\frac{5}{6}} \cdot 5^{\frac{2}{3}} \cdot 7^{\frac{2}{3}}$. Since we are looking for the smallest value of $a + b$, note that the minimal sum given a fixed product is usually given where $a = b$. Since roots are involved, this may not give the actual minimal sum, but we are only after an approximation, so we will assume $a = b$ (see the note below). Thus, we need $a^{\frac{7}{12}} = a^{\frac{1}{4}} \cdot a^{\frac{1}{3}} = 2^{\frac{1}{4}} \cdot 3^{\frac{5}{6}} \cdot 5^{\frac{2}{3}} \cdot 7^{\frac{2}{3}}$, so raising to the $\frac{12}{7}$ yields $a = 2^{\frac{3}{7}} \cdot 3^{\frac{10}{7}} \cdot 5^{\frac{8}{7}} \cdot 7^{\frac{8}{7}} = (2^3 \cdot 3^{10} \cdot 5^8 \cdot 7^8)^{\frac{1}{7}} = 3 \cdot 5 \cdot 7 (2^3 \cdot 3^3 \cdot 5 \cdot 7)^{\frac{1}{7}} = 105 \cdot 7560^{\frac{1}{7}}$. Now, since $3^7 < 7560 < 4^7$, $3 < 7560^{\frac{1}{7}} < 4 \Rightarrow 315 < 105 \cdot 7560^{\frac{1}{7}} < 420 \Rightarrow 315 < a < 420$. It follows that $630 < a + b < 840$, so the sum must be in the interval $[500, 1000)$.

(Note: The calculator's value of this a to four decimal places is 376.0658, so $a + b \approx 752.1316$. If a and b were restricted to be integers, then $a = 18$ and $b = 3675$ makes the expression equal to 630. Using calculus to find the minimum of the function $f(a) = a + b = a +$

$$\left(\frac{2^{\frac{1}{4}} \cdot 3^{\frac{5}{6}} \cdot 5^{\frac{2}{3}} \cdot 7^{\frac{2}{3}}}{a^{\frac{1}{4}}} \right)^3, \text{ one can find the minimal sum to be given by } a + b \approx 319.0585 + 425.4112$$

$= 744.4697$, so assuming a and b are equal is a very good approximation to the minimal sum.)

30. Setting $a = 9^x - 27$ and $b = 27^x - 9$, this equation becomes $a^3 + b^3 = (a + b)^3 = a^3 + b^3$

$+ 3ab(a + b) \Rightarrow 0 = 3ab(a + b)$, so either $a = 0 \Rightarrow x = \frac{3}{2}$, or $b = 0 \Rightarrow x = \frac{2}{3}$, or $a + b = 0$

$9^x + 27^x - 36 = 0 \Rightarrow 9^x + 27^x = 36$. Clearly, $x = 1$ is a solution to this equation, but since 9^x and 27^x both only increase as x increases, this is the only solution to this equation. Therefore, the

sum of the solutions is $\frac{3}{2} + \frac{2}{3} + 1 = \frac{19}{6}$.