

Answers:

<u>Relay 1</u> 1. -1 2. 6 3. 120 4. 361 5. 381 6. 72,390	<u>Relay 4</u> 1. 22 2. 32 3. 5 4. 52 5. 6545 6. 16
<u>Relay 2</u> 1. 1360 2. 136 3. -9 4. 14 5. 13 6. -21	<u>Relay 5</u> 1. 160 2. -5 3. 13 4. 184 5. 4 6. -16
<u>Relay 3</u> 1. 85 2. 4 3. 330 4. 15 5. -420 6. 3031	

Solutions:

Relay 1

1. The vertex for this parabola occurs at $x = -\frac{-3}{2 \cdot 2} = \frac{3}{4}$, so the y -coordinate is $y = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + \frac{1}{8} = \frac{9}{8} - \frac{9}{4} + \frac{1}{8} = -1$.

2. The distance between $(-1, 5)$ and $(-3, -1)$ is $d = \sqrt{(-1 - (-3))^2 + (5 - (-1))^2} = \sqrt{2^2 + 6^2} = \sqrt{4 + 36} = \sqrt{40} \approx 6.3...$, so $\lfloor d \rfloor = 6$.

3. The number of degrees in one interior angle of a regular hexagon is $\frac{180(6-2)}{6} = \frac{720}{6} = 120$.

4. $\frac{x^3 - 12x^2 + 120x - 64}{x - 5} = x^2 - 7x + 85 + \frac{361}{x - 5}$, so the remainder is 361.

5. Since the prime factorization of 361 is 19^2 , the sum of the positive integral factors of 361 is $1 + 19 + 19^2 = 381$.

6. The number of ways to select two distinct marbles from a collection of 381 distinct marbles is $\binom{381}{2} = \frac{381 \cdot 380}{2 \cdot 1} = 72,390$.

Relay 2

1. For her work, Sophia will earn $\frac{\$17}{\cancel{\text{hr}}} \cdot \frac{4 \cancel{\text{hr}}}{\cancel{\text{day}}} \cdot \frac{5 \cancel{\text{days}}}{\cancel{\text{week}}} \cdot \frac{4 \cancel{\text{weeks}}}{1} = \1360 .

2. $\frac{1360}{1+3+6+0} = \frac{1360}{10} = 136$

3. $136 = \begin{vmatrix} 13 & x \\ 5 & 7 \end{vmatrix} = 13 \cdot 7 - 5x = 91 - 5x \Rightarrow -5x = 45 \Rightarrow x = -9$

$$4. \frac{n(n-3)}{2} = (-9)^2 + (-9) + 5 = 77 \Rightarrow n^2 - 3n = 154 \Rightarrow 0 = n^2 - 3n - 154 = (n-14)(n+11)$$

$\Rightarrow n = 14$ (since n is a number of sides, $n > 0$)

5. In order for this system to have either no solutions or an infinite number of solutions, the

coefficient matrix must have determinant equal to 0. Therefore, $0 = \begin{vmatrix} 1 & -14 & C \\ -3 & 4 & 14 \\ C & 1 & 1 \end{vmatrix} = 4 - 196C$

$-3C - 4C^2 - 14 - 42 = -4C^2 - 199C - 52$, and since the discriminant of this quadratic expression is $(-199)^2 - 4(-4)(-52) > 0$, both solutions to this equation are real. Using Vieta's formula, the product of the roots is $\frac{-52}{-4} = 13$.

$$6. \text{ This function is } f(x) = \begin{cases} \frac{-2x^2 + 13x + D}{x-5}, & \text{if } x < 13 \\ \frac{3x^2 - Dx + 22}{x+2}, & \text{if } x > 13 \end{cases}, \text{ so } 1 = f(2) = \frac{-2(2)^2 + 13(2) + D}{2-5}$$

$$= \frac{-8 + 26 + D}{-3} = \frac{18 + D}{-3} \Rightarrow 18 + D = -3 \Rightarrow D = -21.$$

Relay 3

1. The third side x satisfies the inequality $48 = 91 - 43 < x < 91 + 43 = 134$, so any integer from 49 to 133, inclusive will work—there are $133 - 49 + 1 = 85$ such integers.

2. There are two primitive Pythagorean triples with hypotenuse of length 85: 13-84-85 and 36-77-85. Since $85 = 5 \cdot 17$ is the prime factorization of 85, any primitive triple whose hypotenuse length is 5 or 17 can be scaled up to make a triple with hypotenuse of length 85. There is one for each of these: 3-4-5 and 8-15-17. Therefore, there are 4 such triples (just to have them listed, they are 13-84-85, 36-77-85, 51-68-85, and 40-75-85).

3. The length of the hypotenuse is $4^3 - 3 = 61$, and since 61 is prime, this can only be a primitive Pythagorean triple, and this triple is 11-60-61. Therefore, the area enclosed by this triangle is

$$\frac{1}{2} \cdot 11 \cdot 60 = 330.$$

4. To make this sum as small as possible, we need to find the first instances, starting at the top, of 330 in Pascal's triangle. The first such instances occur in the twelfth row as the fifth and

eight entries. Therefore, $\binom{11}{4} = 330 = \binom{11}{7}$. There are no other occurrences of 330 through

the sixteenth row, so for any other occurrence of 330 in Pascal's triangle, $n > 15$. Therefore, the least possible value of $n + k$ is $11 + 4 = 15$.

5. The fourteenth term of $(2x - 1)^{15}$, written in order of decreasing power of x , would be

$$\binom{15}{13}(2x)^2(-1)^{13} = \frac{15 \cdot 14}{2 \cdot 1} \cdot 2^2 \cdot (-1) \cdot x^2, \text{ so the coefficient is } \frac{15 \cdot 14}{2 \cdot 1} \cdot 2^2 \cdot (-1) = -420.$$

6. Since this is an arithmetic series, with the given information, letting n be the number of

terms, we have $\frac{n}{2}(-420 + K) = 1 - 2(-420) = 841 \Rightarrow n = \frac{2 \cdot 841}{-420 + K} = \frac{1682}{-420 + K}$, so because n must

be a positive integer, $K - 420$ must be positive and divide $1682 = 2 \cdot 29^2$. The positive divisors of 1682 are 1, 2, 29, 58, 841, and 1682, so setting $K - 420$ equal to each of these and solving for K yields $K = 421, 422, 449, 478, 1261$, and 2102 . However, 2102 does not work in this problem since this would give $n = 1$, and impossibility since $T = 1 - 2T$ does not yield $T = -420$.

Therefore, the sum of all possible values of K is $421 + 422 + 449 + 478 + 1261 = 3031$.

Relay 4

1. This is the length of the major axis, which equals $2a$, where a^2 is the larger denominator when the ellipse is written in this form. Therefore, the distance is $2\sqrt{121} = 2 \cdot 11 = 22$.

2. In a similar famous problem, when only considering 365 birthdays, 23 people is the minimum number for this probability to be above 50%. Therefore, if we increase the number of options of birthday while decreasing the number of people (366 birthdays, 22 people), the probability will decrease. Therefore, $p < \frac{1}{2}$, making the answer to this question 32.

3. $\log_2 32 = 5$

4. The triangle's given sides have lengths $5^2 = 25$ and $6^2 = 36$, so the length of the angle

bisector from this vertex to the third side of length c is $\sqrt{25(36)\left(1 - \frac{c^2}{(25+36)^2}\right)}$
 $= 5(6)\sqrt{1 - \frac{c^2}{61^2}} = 30\sqrt{1 - \frac{c^2}{61^2}}$. Therefore, $30\sqrt{1 - \frac{c^2}{61^2}} > 15 \Rightarrow \sqrt{1 - \frac{c^2}{61^2}} > \frac{1}{2} \Rightarrow 1 - \frac{c^2}{61^2} > \frac{1}{4}$
 $\Rightarrow \frac{c^2}{61^2} < \frac{3}{4} \Rightarrow c^2 < \frac{3}{4} \cdot 61^2 = \frac{11163}{4} = 2790.75$. Since $52^2 = 2704$ and $53^2 = 2809$, $c < \sqrt{2790.75}$
 $\approx 52.8\dots$, so the greatest possible integral length of the third side is 52.

5. Go ahead and give five candies to each of the four people, leaving 32 candies to distribute. Using the stars and bars method, the number of ways of distributing these remaining candies is

$$\binom{4+32-1}{32} = \binom{35}{32} = \frac{35 \cdot 34 \cdot 33}{3 \cdot 2 \cdot 1} = 6545.$$

6. $6545 = 5^1 \cdot 7^1 \cdot 11^1 \cdot 17^1$ is the prime factorization of 6545, so the number of positive integral factors of 6545 is $(1+1)(1+1)(1+1)(1+1) = 16$.

Relay 5

1. Since the common ratio is $\frac{\frac{128}{5}}{32} = \frac{4}{5}$, the sum of the infinite geometric series is $\frac{\frac{32}{1 - \frac{4}{5}}}{\frac{1}{5}}$
 $= \frac{32}{\frac{1}{5}} = 32 \cdot 5 = 160$.

2. The function is $y = \frac{3x^2 + 17x - 160}{x^2 + x - 20} = \frac{(3x + 40)(x - 4)}{(x + 5)(x - 4)}$, so the vertical asymptote is $x = -5$.

Therefore, $A = -5$.

3. This number is $(2(-5)^2 - (-5))! = 55!$, and the number of consecutive zeros at the end equals the number of factors of 10 in the factorization of $55!$, and this is limited by the number of factors of 5 in the factorization of $55!$ (there are many more factors of 2 in this factorization). There are eleven multiples of 5 in this factorization, but two of those multiples, 25 and 50, have

an additional factor of 5. Therefore, there are a total of 13 factors of five, meaning there are 13 consecutive zeros at the end of $55!$.

4. Since 13 is prime, the answer to this question is 8.

5. This equation is $4^x - 2^{x+1} - 40 = 184 \Rightarrow 0 = 4^x - 2^{x+1} - 224 = (2^x)^2 - 2 \cdot 2^x - 224$
 $= (2^x + 14)(2^x - 16) \Rightarrow 2^x = 16 \Rightarrow x = 4$ ($2^x = -14$ has no real solutions).

6. The two line equations are $(4^3 - 4 + 1)x + (2 \cdot 4)y = -287 \Rightarrow 61x + 8y = -287$ and
 $(1 - 2 \cdot 4)x + (3 \cdot 4 + 2)y = -161 \Rightarrow -7x + 14y = -161 \Rightarrow x - 2y = 23 \Rightarrow 4x - 8y = 92$. Adding the two sides of these two equations together yields $65x = -195 \Rightarrow x = -3 \Rightarrow y = -13$, so the intersection is $(-3, -13)$, meaning $A + B = (-3) + (-13) = -16$.