

Answer Key:

1. C
2. A
3. D
4. B
5. B

6. D
7. D
8. A
9. A
10. E

11. D
12. B
13. B
14. C
15. D

16. C
17. A
18. C
19. B
20. C

21. A
22. B
23. A
24. E
25. C

26. B
27. D
28. B
29. D
30. B

1. **C:** $\sum_{n=0}^{2026} \left(\frac{n}{1013} - 1 \right) = \frac{2026(2027)}{2(1013)} - 2027 = 2027 - 2027 = 0.$

2. **A:** Recall that $\log_b a = \frac{\log a}{\log b}$, so $\prod_{n=2}^{31} \log_n(n+1) = \prod_{n=2}^{31} \frac{\log(n+1)}{\log n} = \frac{\log 3}{\log 2} * \frac{\log 4}{\log 3} * \frac{\log 5}{\log 4} * \dots * \frac{\log 32}{\log 31} = \frac{\log 32}{\log 2} = \log_2 32 = 5.$

3. **D:** $a_1 = 2(1) - 4 = -2$. $a_2 = 2(2) - (-2) = 6$. $a_3 = 2(3) - 6 = 0$. $a_4 = 2(4) - 0 = 8$. $a_5 = 2(5) - 8 = 2$.

4. **B:** If $a_4 = a_5$, then $a_5 = 2(5) - a_5 \Rightarrow a_5 = 5 = a_4$. Then, $a_4 = 5 = 2(4) - a_3 \Rightarrow a_3 = 3 = 2(3) - a_2 \Rightarrow a_2 = 3 = 2(2) - a_1 \Rightarrow a_1 = 1 = 2(1) - a_0 \Rightarrow a_0 = 1$

5. **B:** This asks for the total circumference of all the circles. $C = 2\pi \sum_{r=0}^{\infty} r_n = 2\pi \sum_{r=0}^{\infty} 8 \left(\frac{1}{3} \right)^n = 16\pi \left(\frac{1}{1-\frac{1}{3}} \right) = 16\pi \left(\frac{3}{2} \right) = 24\pi$

6. **D:** The total area is $A = \pi \sum_{r=0}^{\infty} r_n^2 = \pi \sum_{r=0}^{\infty} 64 \left(\frac{1}{9^n} \right) = 64\pi \left(\frac{1}{1-\frac{1}{9}} \right) = 64\pi \left(\frac{9}{8} \right) = 72\pi$

7. **C:** We want the point to lie have smallest radius $r_0, r_2, \dots$. This corresponds to any of within r_0 but outside r_1 , within r_2 but outside r_3 , and in general, within r_{2k} but outside r_{2k+1} . The area for each region is $R_k = A_{2k} - A_{2k+1} = \pi r_{2k}^2 - \pi r_{2k+1}^2 = 64\pi \left(\frac{1}{9^{2k}} - \frac{1}{9} * \frac{1}{9^{2k}} \right) = 64\pi \left(\frac{8}{9} * \frac{1}{9^{2k}} \right) = \frac{512\pi}{9} \left(\frac{1}{9^{2k}} \right)$ with total area $\sum_{k=0}^{\infty} R_k = \frac{512\pi}{9} \sum_{k=0}^{\infty} \frac{1}{9^{2k}} = \frac{512\pi}{9} \left(\frac{1}{1-\frac{1}{81}} \right) = \frac{512\pi}{9} \left(\frac{81}{80} \right) = 57.6\pi$. The area of the largest circle is $A_0 = \pi r_0^2 = 64\pi$ for a probability of $\frac{57.6\pi}{64\pi} = \frac{9}{10}$.

8. **A:** $x_n = x_0 + nd$, so $x_2 = x_0 + 2d = 4$ and $x_{20} = x_0 + 26d = 26$. Solving for x_0 and d gives $x_0 = \frac{14}{9}$ and $d = \frac{11}{9}$, so $x_{29} = \frac{14}{9} + 29 * \frac{11}{9} = \frac{333}{9} = 37$.

9. **A:** The primes (in order) are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, **37**

10. **E:** All 4 numbers listed are primes. Their indices are 25, 31, 38, and 44, respectively.

11. **D:** From the previous question, $P_{25} = 97$, so $P_{26} = 101 > 100$. $101 = 26 * 3 + 23$.

12. B: If there are $\frac{N}{\ln N}$ primes less than N , then the $\frac{N}{\ln N}$ th prime should roughly equal N . Thus,
 $a = P_{\frac{n}{\ln n}}$.

13. B: Split into two cases: $n \leq 7$ and $n > 7$. $S_1 = \sum_{n=0}^7 (4n + 3|2n - 14|) = \sum_{n=0}^7 (4n + 3(14 - 2n)) = \sum_{n=0}^7 (42 - 2n) = -2 \left(\frac{7(8)}{2} \right) + 8 * 42$ and $S_2 = \sum_{n=8}^{13} (4n + 3|2n - 14|) = \sum_{n=8}^{13} (4n + 3(2n - 14)) = \sum_{n=8}^{13} (10n - 42) = 10 \left(\frac{13(14)}{2} - \frac{7(8)}{2} \right) - 6 * 42$ for a total of $S_1 + S_2 = -6(7)(8) + 5(13)(14) + 2 * 42 = -336 + 910 + 84 = 658$.

14. C: $\sum_{n=1}^6 a_n = \sum_{n=1}^6 (a_6 + (n - 6)d) = \sum_{n=1}^6 (2 + (n - 6)d) \Rightarrow 12 - 15d = 42 \Rightarrow -15d = 30 \Rightarrow d = -2$. $\sum_{n=1}^6 \frac{(-2)^n}{6} = \frac{1 - 2 - (-2)^7}{6 \cdot 1 - (-2)} = 7$.

15. D: Performing long division to get the decimal expansion of $\frac{1}{7} = 0.\overline{142857}$ has period 6. $2026 \equiv 4 \pmod{6}$, so the 2026th digit will be 8.

16. C: Each term is of the form $x_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} * \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{(n+1) - n} = \sqrt{n+1} - \sqrt{n}$.
 $\sum_{n=1}^{2024} x_n = \sum_{n=1}^{2024} (\sqrt{n+1} - \sqrt{n}) = \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \dots + \sqrt{2025} - \sqrt{2024} = \sqrt{2025} - \sqrt{1} = 45 - 1 = 44$.

17. A: The sequence is $a_n = 15, 20, 33, \dots$ with first difference $d_n^{(1)} = 5, 13, \dots$ and second difference $d_n^{(2)} = 8, \dots$. However, because we are told a_n is quadratic, $d_n^{(2)}$ will be constant. Thus, $d_n^{(1)} = 5, 13, 21, 29, \dots$ and so $a_n = 15, 20, 33, 33 + 21 = 54, 54 + 29 = 83$ for a sum of $54 + 83 = 137$.

18. C: Note that $S = \sum_{n=1}^{\infty} n \left(\frac{2}{3} \right)^n = \frac{2}{3} + \frac{8}{9} + \frac{24}{27} + \frac{64}{81} + \dots$, so $\frac{2}{3}S = \frac{4}{9} + \frac{16}{27} + \frac{48}{81} + \dots \Rightarrow S - \frac{2S}{3} = \frac{S}{3} = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \dots = \frac{2/3}{1 - 2/3} = \frac{2/3}{1/3} = 2 \Rightarrow S = 2(3) = 6$.

19. B: $\bar{x} = \frac{1}{9}(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = \frac{1}{9} \left(\frac{9(10)}{2} \right) = 5$. $S_x^2 = \frac{1}{8} \sum_{i=1}^9 (i - 5)^2 = \frac{1}{8}(2(1 + 4 + 9 + 16)) = \frac{1}{4}(30) = \frac{15}{2}$.

20. C: Reformulating the definition of $S_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \Rightarrow (n-1)S_x^2 = \sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + n\bar{x}^2 = \sum_{i=1}^n x_i^2 - 2n\bar{x} \left(\frac{\sum_{i=1}^n x_i}{n} \right) + n\bar{x}^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2 \Rightarrow 15(12) = 228 - 16(\bar{x}^2) \Rightarrow \bar{x}^2 = \frac{228 - 180}{16} = \frac{48}{16} = 3 \Rightarrow |\bar{x}| = \sqrt{3}$.

21. A: Let $S = 2 + \frac{1}{4 + \frac{1}{2 + \frac{1}{4 + \frac{1}{\dots}}}} \Rightarrow S = 2 + \frac{1}{4 + \frac{1}{S}} = 2 + \frac{S}{4S+1} = \frac{8S+2+S}{4S+1} \Rightarrow 4S^2 + S = 9S + 2 \Rightarrow 4S^2 -$

$8S - 2 = 0 \Rightarrow S = 1 \pm \frac{\sqrt{6}}{2}$, however $S > 0$, so $1 - \frac{\sqrt{6}}{2}$ is extraneous and thus $S = 1 + \frac{\sqrt{6}}{2}$. Note that we are actually solving for $2S - 2 = 2 \left(1 + \frac{\sqrt{6}}{2}\right) - 2 = \sqrt{6}$.

22. B: $G_n = G_0 r^n$, so $G_2 = G_0 r^2 = 12$ and $G_5 = G_0 r^5 = 4$. Dividing $\frac{G_5}{G_2} = \frac{12}{4} = 3 = r^{-3} \Rightarrow r = \frac{1}{\sqrt[3]{3}}$. Then, $G_0 = \frac{12}{r^2} = 12\sqrt[3]{9}$. Thus, $\sum_{n=1}^{\infty} G_{5n} = \sum_{n=1}^{\infty} G_0 r^{5n} = G_0 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{\frac{5n}{3}} = 12\sqrt[3]{9} \left(\frac{3^{-\frac{5}{3}}}{1-3^{-\frac{5}{3}}}\right) = \frac{36}{9-3\sqrt[3]{3}}$.

23. A: $\frac{n+2}{n-1} * \binom{n+1}{3} = \frac{n+2}{n-1} * \left(\frac{(n+1)(n)(n-1)}{6}\right) = \frac{1}{6}(n+2)(n+1)(n)$, so the summation is $\frac{1}{6} \sum_{n=5}^{24} n(n+1)(n+2) \stackrel{n \rightarrow n-1}{=} \frac{1}{6} \sum_{n=6}^{25} n(n+1)(n-1) = \frac{1}{6} \sum_{n=6}^{25} n(n^2 - 1) = \frac{1}{6} \sum_{n=6}^{25} (n^3 - n) = \frac{1}{6} \left(\left[\left(\frac{25(26)}{2}\right)^2 - \left(\frac{25(26)}{2}\right) \right] - \left[\left(\frac{5(6)}{2}\right)^2 - \left(\frac{5(6)}{2}\right) \right] \right) = \frac{1}{6} ([325^2 - 325] - [15^2 - 15]) = \frac{105300-210}{6} = 17515$.

24. E: The common difference in the series is clear 0.4, so the series has explicit form $a_n = -16.4 + 0.4n$. $a_n > 2026 \Rightarrow -16.4 + 0.4n > 2026 \Rightarrow 0.4n > 2042.4 \Rightarrow n > 5106 \Rightarrow n = 5107$ for a sum of $5 + 1 + 0 + 7 = 13$.

25. C: $2026 + 202.\bar{6} + 20.\bar{26} + 2.0\bar{26} = 622 + \left(202 + \frac{6}{9}\right) + \left(20 + \frac{26}{99}\right) + \left(2 + \frac{1}{10} * \frac{26}{99}\right) = 224 + \frac{6*110+26*10+26}{990} = 224 + \frac{946}{990} = 224 + \frac{43}{45}$ with fractional part $\frac{43}{45}$.

26. B: Evaluating the inner product in terms of n , we get that $i_n = \prod_{m=1}^n i^m = i^{\sum_{m=1}^n m} = i^{\left(\frac{n(n+1)}{2}\right)}$ which takes values according to $i_n = i, -i, -1, -1, -i, i, 1, 1 \dots$. The inner product is cyclic with period 8, so

$$\begin{aligned} 34 \sum_{n=1}^{\infty} 2^{-n} i_n &= 34 \sum_{k=1}^{\infty} \sum_{j=1}^8 (2^{-8k-j} i_{8k+j}) = 34 \sum_{k=1}^{\infty} \left(2^{-8k} \sum_{j=1}^8 2^{-j} i_j \right) = 34 \left(\sum_{k=1}^{\infty} 2^{-8k} \right) \left(\sum_{j=1}^8 2^{-j} i_j \right) \\ &= 34 \left(\frac{1}{1-2^{-8}} \right) \left(\frac{i}{2} - \frac{i}{4} - \frac{1}{8} - \frac{1}{16} - \frac{i}{32} + \frac{i}{64} + \frac{1}{128} + \frac{1}{256} \right) \\ &= 34 \left(\frac{256}{255} \right) \left(-\frac{45}{256} + \frac{15}{64} i \right) = -6 + 8i \end{aligned}$$

27. D: $72 = 8 * 9 = 2^3 * 3^2$, so a trailing 0 will occur for every 3 2's and 2 3's. $\frac{2026}{2} = 1013$; $\frac{1013}{2} = 506.5$; $\frac{506}{2} = 253$; $\frac{253}{2} = 126.5$; $\frac{126}{2} = 63$; $\frac{63}{2} = 31.5$; $\frac{31}{2} = 15.5$; $\frac{15}{2} = 7.5$; $\frac{7}{2} = 3.5$; $\frac{3}{2} = 1.5$, so there are $1 + 3 + 7 + 15 + 63 + 126 + 506 + 1013 = 1734$ multiples of 2 and thus $\frac{1734}{3} = 578$ potential trailing 0's. Likewise, $\frac{2026}{3} = 675.3$; $\frac{675}{3} = 225$; $\frac{225}{3} = 75$; $\frac{75}{3} = 25$; $\frac{25}{3} = 8.3$; $\frac{8}{3} = 2.6$, so there are $2 + 8 + 25 + 75 + 225 + 675 = 1010$ multiples of 3 and thus $\frac{1010}{2} = 505$ potential trailing 0's. Therefore, there are $\min(578, 505) = 505$ multiples of 72 in 2026! and so there are 505 trailing zero in $2026!_{72}$.

28. B: Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $M \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} P_{n+2} & P_{n+1} \\ P_{n+1} & P_n \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} aP_{n+1} + bP_n & aP_n + bP_{n-1} \\ cP_{n+1} + dP_n & cP_n + dP_{n-1} \end{bmatrix} \Rightarrow \begin{cases} aP_{n+1} + bP_n = P_{n+2} \\ cP_{n+1} + dP_n = P_n \end{cases} \Rightarrow \begin{cases} a = 2, b = 1 \\ c = 1, d = 0 \end{cases}$. Thus, $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.

29. D: Evaluating the first handful of elements in the Pell sequence: $P_n = 0, 1, 2, 5, 12, 29, 70, 169, 408, \dots$ which give rational approximations for $\sqrt{2}$ of $(\sqrt{2})_n = 1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots$ for which $\frac{100}{71}$ is not an element of this sequence.

30. B: Solving for $c_n^2 = a_n^2 + b_n^2 = (2P_n P_{n+1})^2 + (P_{n+1}^2 - P_n^2)^2 = 4P_n^2 P_{n+1}^2 + P_{n+1}^4 - 2P_{n+1}^2 P_n^2 + P_n^4 = P_{n+1}^4 + 2P_{n+1}^2 P_n^2 + P_n^4 = (P_{n+1}^2 + P_n^2)^2 \Rightarrow c_n = \pm(P_{n+1}^2 + P_n^2)$ for which the negative is clearly extraneous, so $c_n = P_{n+1}^2 + P_n^2$. Then, by using the explicit formula for P_n ,

$$P_n^2 = \left(\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right)^2 = \frac{1}{8} \left((1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} - 2(1+\sqrt{2})^n (1-\sqrt{2})^n \right) = \frac{1}{8} \left((3+2\sqrt{2})^n + (3-2\sqrt{2})^n - (-1)^n \right) \text{ and } P_{n+1}^2 = \frac{1}{8} \left((3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1} - (-1)^{n+1} \right), \text{ so}$$

$$\begin{aligned} c_n &= \frac{1}{8} \left((3+2\sqrt{2})^n + (3-2\sqrt{2})^n - (-1)^n + (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1} - (-1)^{n+1} \right) \\ &= \frac{1}{8} \left((3+2\sqrt{2})^n (1+3+2\sqrt{2}) + (3-2\sqrt{2})^n (1+3-2\sqrt{2}) \right) \\ &= \frac{1}{8} \left((3+2\sqrt{2})^n (4+2\sqrt{2}) + (3-2\sqrt{2})^n (4-2\sqrt{2}) \right) \end{aligned}$$

is dominated by the $(3+2\sqrt{2})^n$ term, so $\frac{c_{n+1}}{c_n} = 3+2\sqrt{2}$ as n grows larger and larger.