2009 Alpha School Bowl

1. Let $\boldsymbol{A\sqrt{B}}$ equal the area of a triangle whose sides are 10, 12, and 6.

Let $\boldsymbol{C\pi}$ equal the area of a circle inscribed in an equilateral triangle with side length 6.

Let $\boldsymbol{D}$ be the least integer in the domain of $y = \sqrt{2x^3 + 11x^2 - 7x - 6}$.

Let $\boldsymbol{E}$ be the value of $\left[\log_3 \frac{24}{11}\right]$, where [ ] represents the greatest integer function. Use $log_3 8 \approx 1.9$ and $log_3 11 \approx 2.2$.

Find the sum of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$.

2. Let $\boldsymbol{A}$ the sum of the digits in $10^{2009} - 10^{1009}$.

Let $\boldsymbol{B} = f(1)$, where f is an unknown function, $g(x) = 3x + 2$, and $f(g(x)) = x^2 - x - 3$.

Let $\boldsymbol{C}$ be the determinant of the matrix $\begin{bmatrix} 0 & 1 & 5 \\ 3 & 0 & 12 \\ 2 & -7 & -8 \end{bmatrix}$, when each entry is multiplied by 2.

Find $\boldsymbol{X}$, when $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$ is written in the form $\frac{\boldsymbol{X}}{\boldsymbol{Y}}$, with $\boldsymbol{X}$ and $\boldsymbol{Y}$ relatively prime.

3. Find the positive difference between the minimum and maximum values of the objective function $z = 5x + 7y$ where $x \geq 0$ and $y \geq 0$ subject to the following constraints:

$$2x + 3y \geq 6$$
$$3x - y \leq 15$$
$$-x + y \leq 4$$
$$2x + 5y \leq 27$$

4. Let $\boldsymbol{A}$ be the sum of the integer solutions to $\frac{x^2 + 2x - 8}{x^3 - 32} > 0$ which are less than 5.

Let $\boldsymbol{B}$ be the number of distinct arrangements of ALPHASCHOOLBOWL if the letters in PASCALCALLSBOO are not used.

Let $\boldsymbol{C}$ be the value of $sin(\alpha + \beta)$ if $sin\,\alpha = \frac{3}{5}$ and $sin\,\beta = -\frac{5}{13}$ if $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3}{2}\pi$.

Find the product $\boldsymbol{ABC}$.

5. Let $\frac{\boldsymbol{A}}{\boldsymbol{B}} = \frac{1}{1\cdot2} + \frac{2}{1\cdot2\cdot3} + \frac{3}{1\cdot2\cdot3\cdot4} + \frac{4}{1\cdot2\cdot3\cdot4\cdot5} + \cdots + \frac{9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot20}$, where A and B are relatively prime.

   The arithmetic mean of two positive numbers exceeds their geometric mean by 50. By how much, $\boldsymbol{C}$, does the square root of the larger number exceed the square root of the smaller number?

   If $\frac{1}{9}, \frac{1}{x}$, and $\frac{3}{7}$ form an arithmetic progression in that order, find the value of x, written as $\frac{\boldsymbol{D}}{\boldsymbol{E}}$, where D and E are relatively prime.

   Find $\boldsymbol{B} - \boldsymbol{A} + \boldsymbol{C} + \boldsymbol{D} + \boldsymbol{E}$.

6. Let $\boldsymbol{A}$ = the sum of the reciprocals of the positive integral divisors of 2010.

   The Brady family has 6 kids: 3 boys and 3 girls. Let $\boldsymbol{B}$ = the number of ways the 6 kids can be seated in a row of 6 chairs, if all 3 boys and all 3 girls are not seated together.

   Find the sum of the digits in the answers to $\boldsymbol{A}$ and $\boldsymbol{B}$.

7. Let $\boldsymbol{A}$= the sum of the maximum value and period of $y = -41\cos 3x + 8$ (use degrees).

   Let $\boldsymbol{B}$ = the sum of the maximum value and period of $y = 3\cos 2x - 4\sin 2x$ (use degrees).

   Let $\boldsymbol{C}$ = the area of triangle PQR if PR = 13, PQ = 15, and $m\angle P = 135^\circ$

   Let $\boldsymbol{D}$ = the square of the distance between J and L if vector OJ has the origin as its initial point and 150° as its direction angle, vector OL has the origin as its initial point and 225° as its direction angle, and. $|\overrightarrow{OJ}| = |\overrightarrow{OL}| = 8$

   Find $\boldsymbol{B} - \boldsymbol{A} + \boldsymbol{C} + \boldsymbol{D}$, written as a fraction.

8. The quotient and difference of two positive numbers are both equal to 9. What is the most common digit $\boldsymbol{A}$ used in the larger number, when written in simplest form as an improper fraction?

   Let $\boldsymbol{B}$ = the 200th triangular number.

   Let $\boldsymbol{C}$ the sum of the first three perfect numbers.

   Let $\boldsymbol{D}$ = the number of integral factors of the lowest common multiple of 205, 408, and 45.

Find $\boldsymbol{A + B + C + D}$.

9. Let $\boldsymbol{A}$ = the range of the members of the solution set for $\sin 5x + \sin x = 0$, where $0^\circ \le x < 360^\circ$.

Let $\boldsymbol{B}$ = the solution set for $\frac{1+\cos\theta}{\sin\theta} = -1$, where $-180^\circ \le \theta \le 180^\circ$.

Let $\boldsymbol{C}$ = the solution set for $\tan\theta - \tan 10^\circ = 1 + \tan\theta\tan 10^\circ$, where $-180^\circ \le \theta \le 180^\circ$.

Let $[\boldsymbol{D}, \boldsymbol{E}] \cup [\boldsymbol{F}, \boldsymbol{G}]$ be the solution set for $2\sin^2 x + \cos x \ge 1$, where $0^\circ \le x \le 360^\circ$.

Find $\boldsymbol{A + B + C + E}$.

10. Given: $4y^2 - 9x^2 + 16y + 18x = 29$.

Let $\boldsymbol{A}$ = eccentricity

Let $\boldsymbol{B}$ = distance between foci

Let $\boldsymbol{C}$ = distance between vertices

Let $\boldsymbol{D}$ = distance between focus with positive ordinate and asymptote with positive slope

Let $\boldsymbol{E}$ = length of transverse axis

Let $\boldsymbol{F}$ = focal width

Find the value of $\frac{\boldsymbol{ABC}}{\boldsymbol{DEF}}$.