Mu Alpha Theta National Convention: Denver 2001
Gemini Test – Alpha Division – Solutions

(#1) If A = Amanda's rate in houses/hr, etc., then we have the following system of equations:

$$\begin{array}{ccccccccc} A & + & J & + & L & & & = & \frac{1}{4} \\ A & + & J & & & + & R & = & \frac{1}{5} \\ A & & & + & L & + & R & = & \frac{1}{6} \\ & & J & + & L & + & R & = & \frac{1}{7} \\ \hline 3A & + & 3J & + & 3L & + & 3R & = & \frac{319}{420} \end{array}$$

By adding them all together and then dividing by 3, we have $A + J + L + R = \frac{319}{1260}$ hours/hr which is their combined rate. Thus, it will take them $\frac{1260}{319}$ hours to complete one house. The correct answer is (B).

(#2) From competitions, we recall that the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$.

Thus, we have $\frac{s^2\sqrt{3}}{4} = 1 \Rightarrow s^2 = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$. But s^2 is the area of that square. The correct answer is (B).

(#3) Using a POLY solver or synthetic division, we obtain $x = 2,\ -3 \pm 2i$. Thus, we have $|-3 + 2i - (-3 - 2i)| = |0 + 4i| = \sqrt{0^2 + 4^2} = 4$. The correct answer is (B).

(#4) We need only find the probabilities of the second and third card matching the suit of the first.
$P\,(first\ card\ is\ anything)\,P\,(second\ matches\ suit\ of\ first)\,P\,(third\ matches\ suit\ of\ first) =$
$\left(\frac{52}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right) = \frac{22}{425}$. The correct answer is (B).

(#5) $\sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{10}\right) = \sin(\alpha + \beta) = \sin\left(\frac{2\pi}{5} + \frac{\pi}{10}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

$1 + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = 1 + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{7}{4}$. The correct answer is (C).

(#6) Row operations for determinants. If we subtract the third row from the first row, we obtain $\begin{vmatrix} x & 2x & 0 \\ x & 2x+2 & 0 \\ 12 & 22 & 1 \end{vmatrix}$. If we subtract the first row from the second row, we obtain $\begin{vmatrix} x & 2x & 0 \\ 0 & 2 & 0 \\ 12 & 22 & 1 \end{vmatrix}$.
By expansion by minors along the third column, this determinant is reduced to
$(1)\begin{vmatrix} x & 2x \\ 0 & 2 \end{vmatrix} = 2x$. Thus, we have $2x = 12 \Rightarrow x = 6$. The correct answer is (D).

(#7) Plug into the standard form three times, since I don't know the formula for finding the radius of the circumscribed circle...
$(1 - h)^2 + (-2 - k)^2 = R^2$
$(-4 - h)^2 + (3 - k)^2 = R^2$
$(7 - h)^2 + (14 - k)^2 = R^2$

$$\left\{\begin{array}{l}(1-h)^2+(-2-k)^2=R^2\\(-4-h)^2+(3-k)^2=R^2\\(7-h)^2+(14-k)^2=R^2\end{array}\right\}\Rightarrow\left\{\begin{array}{l}(1-h)^2+(-2-k)^2=(-4-h)^2+(3-k)^2\\5-2h+h^2+4k+k^2=25+8h+h^2-6k+k^2\\-10h+10k=20\Rightarrow -h+k=2\end{array}\right.\quad \text{AND}$$

$$\left.\begin{array}{l}(-4-h)^2+(3-k)^2=(7-h)^2+(14-k)^2\\25+8h+h^2-6k+k^2=245-14h+h^2-28k+k^2\\22h+22k=220\Rightarrow h+k=10\end{array}\right\}\Rightarrow\left\{\begin{array}{c}-h+k=2\\h+k=10\end{array}\right\}\Rightarrow$$

$h=4,\ k=6.$

This gives us $R^2=(1-4)^2+(-2-6)^2=73$, so the area must be 73π.

The correct answer is (A).

(#8) $\left(\frac{3}{5}\right)(185)=111$ drinking whales.

$12+8+x+5x+1=111\Rightarrow x=15$. The correct answer is (D).

(#9) By similar triangles (or the "chordal product theorem"), we have $(AE)(EB)=(CE)(ED)$.

If $x=CE$, then we have $(6)(4)=x(ED)\Rightarrow ED=\frac{24}{x}\Rightarrow x+\frac{24}{x}=14\Rightarrow x=2$ or 12.

For convenience, we choose $x=CE=2$. Now I must find the location of the circumcenter of triangle ABC. Through analytic geometry, we assign $A(-5,0)$ and $B(5,0)$ which puts them 10 units apart. The perpendicular bisector of AB is the y-axis. Since C is six units east and two units north of A, the coordinate of C must be $(1,2)$.

The midpoint of AC is $\left(\frac{-5+1}{2},\frac{0+2}{2}\right)=(-2,1)$. The slope of AC is clearly $\frac{1}{3}$, so the slope of the perpendicular bisector must be the negative reciprocal, (-3). So the equation of the perpendicular bisector of AC is $(y-1)=-3(x-(-2))\Rightarrow y=-3x-5$. The y-intercept is (-5) and so both (all) perpendicular bisectors must intersect at $O(0,-5)$.

$OA=\sqrt{(0-(-5))^2+(-5-0)^2}=\sqrt{50}=r$. The area of the circumscribed circle is 50π. The correct answer is (B).

(#10) I prefer to use the (calculator) cross product for this one. If $A(1,2,3)$, $B(4,6,15)$, and $C(6,14,87)$ are the vertices, then $\overrightarrow{AB}=\langle 3,4,12\rangle$ and $\overrightarrow{AC}=\langle 5,12,84\rangle$. The length of the cross product vector will be the area of the parallelogram created by vector addition, so the area of the triangle is one-half this value.

$\langle 3,4,12\rangle\times\langle 5,12,84\rangle=\langle 192,-192,16\rangle$. $\|\langle 192,-192,16\rangle\|=\sqrt{192^2+(-192)^2+16^2}=272$.

One-half of 272 is 136. The correct answer is (D).

(#11) By algorithmic multiplication, the correct answer is (E).

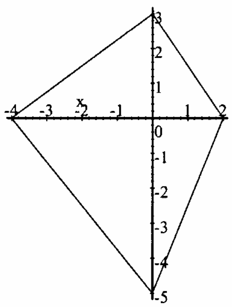
$$\begin{array}{cccccccccccc}
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1\\
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1_7
\end{array}$$

(#12) If x = the number of seats in the smaller sections, then we have $x + x + kx = 150$ where k is a natural number greater than 1 and x must be a natural number. Thus, we have $(k + 2)\,x = 150$. We want x to be as large as possible which will make the larger section as small as possible. Since $k = 2$ yields no solution, we choose $k = 3$ and $x = 30$. Thus, the two smaller sections have 30 seats and the larger section has 90 seats. The correct answer is (B).

(#13) The binary representation of 2001 is 11111010001_2. The sum of the digits is 7.

The correct answer is (D).

(#14) By trial-and-error, we can easily find the appropriate sums of squares to match the given hypotenuse values.



The diagonal pieces should be 2, 3, 4, and 5. Start in the positive-x direction and go around counterclockwise.

$\sqrt{2^2+3^2} = \sqrt{13} \quad \sqrt{3^2+4^2} = 5$
$\sqrt{4^2+5^2} = \sqrt{41} \quad \sqrt{5^2+2^2} = \sqrt{29}$

The area of any quadrilateral with perpendicular diagonals is one-half the product of the diagonals.
$\frac{1}{2}\,(6)\,(8) = 24$ square units. The correct answer is (A).

(#15) The hexagon with the largest area is regular, so we find the area of six equilateral triangles with side 6. $6\left(\frac{6^2\sqrt{3}}{4}\right) = 54\sqrt{3}$.

We can flatten the hexagons and make their areas arbitrarily close to zero (non-degenerate), so the range of areas must be $\left(0, 54\sqrt{3}\right]$. The correct answer is (C).

(#16) $f\,(g\,(x)) = f\,(cx + d) = a\,(cx + d) + b = acx + ad + b$.

$g\,(f\,(x)) = g\,(ax + b) = c\,(ax + b) + d = acx + bc + d$.

If they are equal, then $acx + ad + b = acx + bc + d \Rightarrow ad + b = bc + d \Rightarrow ad - d = bc - b \Rightarrow$

$d\,(a - 1) = b\,(c - 1)$. As long as $b \neq 0$ and $a \neq 1$, we can divide both sides by $b\,(a - 1)$ and obtain the relationship $\frac{c-1}{a-1} = \frac{d}{b}$. If this relationship exists, then $f\,(g\,(x)) = g\,(f\,(x))$.

The correct answer is (B). The other three choices will not guarantee $f\,(g\,(x)) = g\,(f\,(x))$.

(#17) On a TI-85/86, we have **sum seq**$((x + 1)\,(x + 2)\,, x, 1, 20, 1)$. This gives us 3,540.

The correct answer is (A).

(#18) Using the standard labeling, we have $a^2 = b^2 + c^2 - 2bc\cos(A) \Rightarrow$

$a = \sqrt{7^2 + 5^2 - 2\,(7)\,(5)\cos(19°)} \doteq 2.795$.

Since $c = 5$ is not the longest side, it must be opposite an acute angle.

By the Law of Sines, we have $\frac{\sin(C)}{c} = \frac{\sin(A)}{a} \Rightarrow \sin(C) = \frac{5\sin(19°)}{\sqrt{74 - 70\cos(19°)}} \Rightarrow$

$C = \arcsin\left(\frac{5\sin(19°)}{\sqrt{74-70\cos(19°)}}\right) \doteq 35.6°$.

This leaves $B = 180° - 19° - 35.6° = 125.4°$. The correct answer is (A).

(#19) $\log_3\left(x^4\right)+\log_x\left(3^{21}\right)=20 \Rightarrow 4\log_3(x)+21\log_x(3)=20 \Rightarrow 4\left(\frac{\ln(x)}{\ln(3)}\right)+21\left(\frac{\ln(3)}{\ln(x)}\right)=20.$

Let $u=\ln(x) \Rightarrow 4u^2+21(\ln(3))^2=20\ln(3)\,u \Rightarrow$
$4u^2-20\ln(3)\,u+21(\ln(3))^2=(2u-3\ln(3))(2u-7\ln(3))=0 \Rightarrow u=\ln(x)=\frac{3}{2}\ln(3)\,,\ \frac{7}{2}\ln(3)\,.$
$x=3^{3/2},\ 3^{7/2}=3\sqrt{3},\ 27\sqrt{3}.$ The sum of the roots is $30\sqrt{3}$. The correct answer is (D).

(#20) We pair up the nontrivial proper integral factors.
$(2)\left(\frac{108}{2}\right)(3)\left(\frac{108}{3}\right)(4)\left(\frac{108}{4}\right)(6)\left(\frac{108}{6}\right)(9)\left(\frac{108}{9}\right)=108^5.$ The correct answer is (D).

(#21) Let $x=\sqrt{\frac{15}{2}+\frac{1}{2}\sqrt{\frac{15}{2}+\frac{1}{2}\sqrt{\frac{15}{2}+\ldots}}} \Rightarrow x=\sqrt{\frac{15}{2}+\frac{1}{2}x} \Rightarrow x^2=\frac{15}{2}+\frac{1}{2}x \Rightarrow x=-\frac{5}{2},\ 3.$

Clearly, we must have $x=3$. The correct answer is (B).

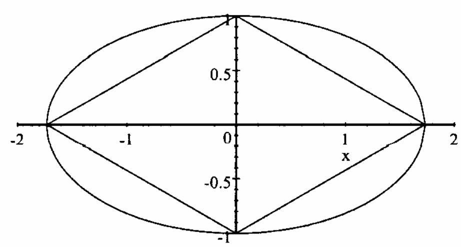
(#22) The three sectors inside the equilateral triangle form a semicircle $(3\times 60^\circ=180^\circ)$ whose area is $\frac{\pi}{2}$. The area of the equilateral triangle is $\frac{(2)^2\sqrt{3}}{4}=\sqrt{3}$.

The difference is inside the triangle, but outside all three circles: $\sqrt{3}-\frac{\pi}{2}$.
The correct answer is (B).

(#23) The mode is 1, of course. The median is $\frac{F_{10}+F_{11}}{2}=72$. It is not difficult to sum them all up, so the mean is 885.5. Since the series resembles a geometric series as $n\to+\infty$, we expect the mean to be much larger than the median since the later terms grow geometrically.

The correct answer is (C).

(#24) The altitude of an equilateral triangle of side length 2 is $\sqrt{3}$.



This is clearly the best configuration.
The semimajor axis is $\sqrt{3}$ and the semiminor axis is 1.
The area is $\pi ab=\sqrt{3}\pi$.
The correct answer is (C).

The equation of the ellipse is $\frac{x^2}{3}+\frac{y^2}{1}=1$.

(#25) Clearly, we will pack a triangle-number of unit circles onto the equilateral triangle. The distance between the centers of the first and last circles on the "bottom" row of circles is $2n$, so there must be $(n+1)$ circles in the bottom row.

(One unit for the end circles and $2(n-1)$ units for the intermediate circles. This adds up to $2n$.)

The formula for the kth triangular number is $\frac{k(k+1)}{2}$, so the $(n+1)$st triangular number is $\frac{(n+1)(n+2)}{2}$.

The correct answer is (B).

(#26) Any proper divisor m which is neither 1, nor $\sqrt{N}$ will be paired with its "complementary" divisor $\frac{N}{m}$.

The mate of the number 1 is not proper and the mate of $\sqrt{N}$ is itself, so D, the total number of proper divisors of N must be even.

In mod 13, the number 12 is equivalent to (-1).

Thus, we see that $12^D \equiv (-1)^D \bmod 13 = 1 \bmod 13$. The correct answer is (A).

(#27) Nothing fancy. Just multiply the series by $\frac{2}{3}$ and then subtract. The result is a geometric series.

$$
\begin{array}{ccccccccccc}
S & = & \frac{2}{3} & + & 2\left(\frac{2}{3}\right)^2 & + & 3\left(\frac{2}{3}\right)^3 & + & 4\left(\frac{2}{3}\right)^4 & + & \ldots \\
\frac{2}{3}S & = & & & \left(\frac{2}{3}\right)^2 & + & 2\left(\frac{2}{3}\right)^3 & + & 3\left(\frac{2}{3}\right)^4 & + & \ldots \\
\hline
\frac{1}{3}S & = & \frac{2}{3} & + & \left(\frac{2}{3}\right)^2 & + & \left(\frac{2}{3}\right)^3 & + & \left(\frac{2}{3}\right)^4 & + & \ldots
\end{array}
$$

$\frac{1}{3}S = \dfrac{\frac{2}{3}}{1-\frac{2}{3}} = 2 \Rightarrow S = 6$. The correct answer is (B).

(#28) We assume that the "sloppy" cuts are still basically straight planar cuts. By carefully angling the cuts, we obtain the following sequence (which was still basically impossible for us to draw neatly on a diagram):

Cuts	1	2	3	4	5	6	7	8
Brownies	2	4	7	11	16	22	29	37

The nth cut always creates n new pieces. In practice, it becomes difficult after $n = 6$, but in theory, the process can be replicated ad infinitum. The correct answer is (A).

(#29) We will need the difference of two cubes in order to rationalize the denominator.

$(a-1)\left(a^2+a+1\right) = a^3 - 1$. Let $a = \sqrt[3]{4}$. The trick is to realize that $\sqrt[3]{4}$ is really $\left(\sqrt[3]{2}\right)^2$.

$$
\frac{3\left(\left(\sqrt[3]{4}\right)^2+\sqrt[3]{4}+1\right)}{\left(\sqrt[3]{4}-1\right)\left(\left(\sqrt[3]{4}\right)^2+\sqrt[3]{4}+1\right)} = \frac{3\left(2\sqrt[3]{2}+\sqrt[3]{4}+1\right)}{\left(\sqrt[3]{4}\right)^3-1} = \frac{3\left(\left(\sqrt[3]{2}\right)^2+2\sqrt[3]{2}+1\right)}{3} =
$$

$\left(\sqrt[3]{2}\right)^2 + 2\sqrt[3]{2} + 1 = \left(\sqrt[3]{2}+1\right)^2$. Now we take the square root and the answer is $\sqrt[3]{2}+1$. The correct answer is (B).

(#30) It is fairly easy to find combinations of numbers which will cause choices (A), (B), and (D) to fail. Will choice (C) work? NO.

If $\frac{3\sqrt{abc}}{2} \le \frac{a+b+c}{2}$, then $\sqrt{abc} \le \frac{a+b+c}{3}$. This resembles the Schwarz inequality, but it does not work!

Let $a = b = c = 100$. $\sqrt{100^3} = 1000 \not\le \frac{100+100+100}{3} = 100$. The correct answer is (E).

(#31) We factor.

$$xy^4 + xy^2z^2 - 5y^4 - 5y^2z^2 + 3xy^2 + xz^2 - 15y^2 - 5z^2 + 2x - 10 = 0$$
$$xy^4 - 5y^4 + xy^2z^2 - 5y^2z^2 + 3xy^2 - 15y^2 + xz^2 - 5z^2 + 2x - 10 = 0$$
$$\left(y^4 + y^2z^2 + 3y^2 + z^2 + 2\right)(x-5) = 0$$
$$\left(y^2 + 1\right)\left(y^2 + z^2 + 2\right)(x-5) = 0.$$

We see that $y^2 = -1$ and $y^2 + z^2 = -2$ have no real solutions, so we are left with $x = 5$.

The correct answer is (C).

(#32) $\dfrac{(F_{n+1} + F_n)(F_{n+1} - F_n)}{F_{n+2}} = \dfrac{F_{n+2}(F_{n+1} - F_n)}{F_{n+2}} = F_{n+1} - F_n = F_{n-1}.$

The correct answer is (A).

(#33) Draw the right triangle with $BC = 6$, $CA = 8$, and hypotenuse $AB = 10$. Construct the altitude CE from C to AB. By similar triangles, we have $CE = 4.8$, $BE = 3.6$, and $EA = 6.4$. If we produce the solid of revolution about the hypotenuse, it will be the union of two right circular cones with the same base. Don't forget that the base is *inside* the solid and does not contribute to the surface area!

The total surface area will be $\pi(4.8)(6) + \pi(4.8)(8) = (67.2)\pi = \frac{336}{5}\pi$.

The two other obvious cones of revolution have surface areas $\pi\left(6^2\right) + \pi(6)(10) = 96\pi$ and $\pi\left(8^2\right) + \pi(8)(10) = 144\pi$. The difference between the largest and smallest is

$144\pi - \frac{336}{5}\pi = \frac{384}{5}\pi$. The correct answer is (D).

(#34) $\cos(4x) = \cos(2(2x)) = 1 - 2\sin^2(2x)$.

$1 - 2\sin^2(2x) = 3\sin(2x) + 2 \Rightarrow 0 = 2\sin^2(2x) + 3\sin(2x) + 1 = (2\sin(2x) + 1)(\sin(2x) + 1)$.

Case 1: $2\sin(2x) + 1 = 0 \Rightarrow \sin(2x) = -\frac{1}{2} \Rightarrow 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \Rightarrow x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Case 2: $\sin(2x) + 1 = 0 \Rightarrow \sin(2x) = -1 \Rightarrow 2x = \frac{3\pi}{2}, \frac{7\pi}{2} \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

The sum of all legal solutions is $\frac{7\pi}{12} + \frac{11\pi}{12} + \frac{19\pi}{12} + \frac{23\pi}{12} + \frac{3\pi}{4} + \frac{7\pi}{4} = \frac{15\pi}{2}$.

The correct answer is (E).

(#35) Using the same procedure as in (#10), we assign $C(1,1,0)$, $A(a,2,0)$, and $B(3,b,0)$.

Thus, we have $\overrightarrow{CA} = \langle a-1, 1, 0\rangle$ and $\overrightarrow{CB} = \langle 2, b-1, 0\rangle$ and the length of

$\overrightarrow{CA} \times \overrightarrow{CB} = (a-1)(b-1) - 2 = 12 - 2 = 10.$

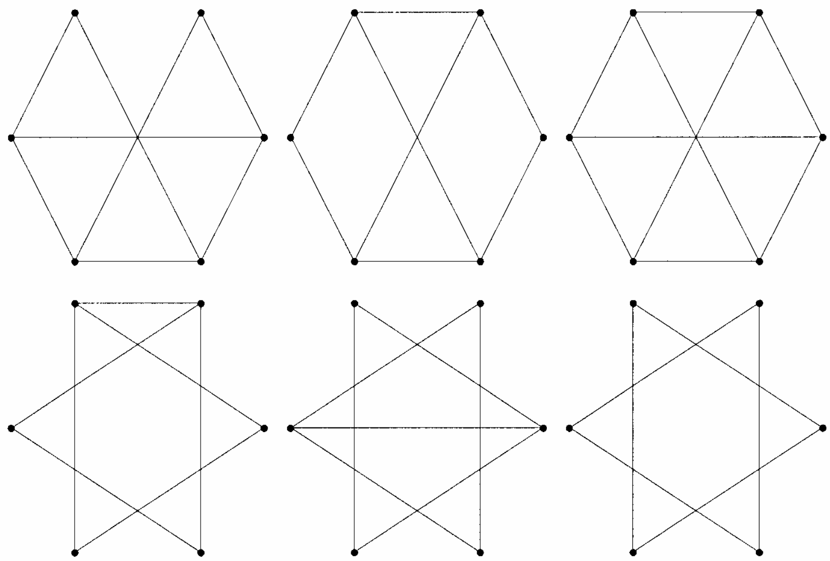
One-half of 10 is 5. The correct answer is (A).

(#36) A graph theory question. Let's say that we will connect two vertices if the pair of crows trust each other. Then we want to check if the graph G contains a triangle (3-cycle) or its complement G^C contains a triangle.

We can easily produce a five-vertex graph which does neither. Let G be a pentagon (5-cycle). Then G^C is also a pentagon and neither contains a triangle. Note that a total of 10 edges were used, 5 in G and 5 in G^C.

In a six-vertex graph, there are $\binom{6}{2} = 15$ possible edges. So fewest number of edges G or G^C could have is 8.

There are only two graphs (up to isomorphism) with 8 edges and one graph with 9 edges which are triangle-free. All of their complements clearly contain a triangle. The three interesting graphs are depicted below (remember that the center of the hexagon is *not* a vertex). The complement of each graph is depicted directly below the original. So it is impossible to construct a six-vertex graph with the desired attributes. The correct answer is (B).



(#37) $P\left(k \,\&\, o \text{ in the same net} \mid \text{each net contains at least one letter}\right) =$

$$\frac{P\left(k \,\&\, o \text{ in the same net } AND \text{ each net contains at least one letter}\right)}{P\left(\text{each net contains at least one letter}\right)}. \quad \text{(Bayes' formula.)}$$

How could we NOT have at least one letter in each net?

$P\left(\text{all the letters are distributed among two nets}\right) = 3\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10}.$

We need the extra factor of '3' for the arbitrary choice of empty net (hmm... it must be third period...).

If the triplet (a, b, c) describes the number of letters in each net, then this probability covers the cases $(0, b, 10-b)$, $(a, 0, 10-a)$, and $(a, 10-a, 0)$. Unfortunately, we have counted $(0, 0, 10)$, $(0, 10, 0)$, and $(10, 0, 0)$ twice, so we must subtract the probability of having two empty nets, $3\left(\frac{1}{3}\right)^{10}$, to obtain

$P\left(\text{at least one net is empty}\right) = 3\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10} - 3\left(\frac{1}{3}\right)^{10} = \frac{341}{6561}.$

$P\left(\text{each net contains at least one letter}\right) = 1 - \frac{341}{6561} = \frac{6220}{6561}.$

Now comes the tough one. We want to place 'k' and 'o' in the same net but we must also ensure that at least one letter ends up in each of the other two nets.

We will arbitrarily place the two magic letters in net #1 giving us independent factors of $(3)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{3}$.

We violate the condition if the other eight letters are distributed among #1 and #2 and this occurs $\left(\frac{2}{3}\right)^8$ of the time. The same probability applies if we distribute among #1 and #3 only. Unfortunately, we have counted the case where all ten letters end up in #1 twice, so we must subtract $\left(\frac{1}{3}\right)^8$ to eliminate duplication.

$P(k \ \& \ o$ together and NOT distributed properly$) = \frac{1}{3}\left(2\left(\frac{2}{3}\right)^8 - 1\left(\frac{1}{3}\right)^8\right) = \left(\frac{1}{3}\right)\left(\frac{511}{6561}\right).$

$P(k \ \& \ o$ together AND each net contains at least one letter$) = \frac{1}{3}\left(1 - \frac{511}{6561}\right) = \frac{6050}{19683}.$

$P(k \ \& \ o$ in the same net $\mid$ each net contains at least one letter$) = \left(\frac{6050}{19683}\right) / \left(\frac{6220}{6561}\right) = \frac{605}{1866}.$

The correct answer is (E).

(#38) If X is the value of one roll, then $E(X) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}.$

The expected number of trials while waiting for the first occurrence of an event with probability p is $\frac{1}{p}$. (This is a direct result of the negative hypergeometric distribution.) So after the first roll, we are waiting to roll the same number again $\left(p = \frac{1}{6}\right)$, and the expected number of rolls is 6.

Thus, we want the sum of the expectations for all seven rolls. This is $7\left(\frac{7}{2}\right) = \frac{49}{2}.$

The correct answer is (C).

(#39) If $P(n) =$ the probability that the second head occurs on the nth toss, then it is easy to verify these facts:

$P(2) = P(HH) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$

$P(3) = P(THH \text{ or } HTH) = 2\left(\frac{1}{2}\right)^3 = \frac{2}{8}.$

$P(4) = P(TTHH \text{ or } THTH \text{ or } HTTH) = 3\left(\frac{1}{2}\right)^4 = \frac{3}{16}.$

$P(k) = \frac{k-1}{2^k}.$

Since each player flips independently of the other, the probability that they will end up on the same number must be

$$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{2}{8}\right)\left(\frac{2}{8}\right) + \left(\frac{3}{16}\right)\left(\frac{3}{16}\right) + \ldots = \sum_{k=2}^{\infty}\left(\frac{k-1}{2^k}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{k}{2^{k+1}}\right)^2 = \frac{5}{27}.$$

Without using calculus, we prefer the calculator approach and just sum enough terms so that convergence is obvious.

The proof of $\sum_{k=1}^{\infty}\frac{k^2}{n^k} = \frac{n(n+1)}{(n-1)^3}$ takes too much space, so we usually just memorize it.

On a TI-85/86, we have `sum seq`$((x/2 \wedge (x+1)) \wedge 2, x, 1, 20, 1)$ which gives us 0.185185...

This can easily be converted to $\frac{5}{27}$. The correct answer is (D).

(#40) Using your POLY solver or a ton of synthetic division, the roots should be $x = 4$ with multiplicity two and $x = 6$ with multiplicity three. The sum of the distinct roots is 10.

The correct answer is (C).