Individual
Alpha
Test #221

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "E) NOTA" means none of the above answers is correct.

1. What is the magnitude of the vector $\vec{v} = \langle -28, 12, 21 \rangle$?

A) 5 B) 17 C) 28 D) 37 E) NOTA

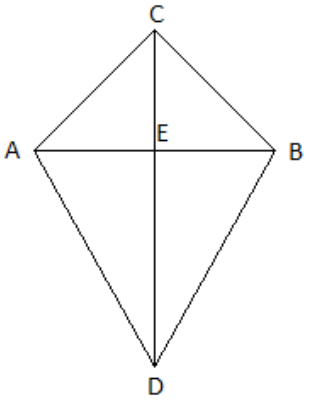
2. Find the value of $\left\lfloor \sum_{n=0}^{5} \sin(15n)^\circ \right\rfloor$, where $\lfloor x \rfloor$ represents the greatest integer $n \le x$.

A) 3 B) 2 C) 5 D) 4 E) NOTA

3. In the Cartesian coordinate plane, how many points of intersection do the graphs of polar equations $r^2 = 4\cos(3\theta)$ and $\tan\theta = \sqrt{3}$ have?

A) 1 B) 2 C) 3 D) 4 E) NOTA

4. In the kite to the right, where $|\overline{AC}| = |\overline{BC}|$, $|\overline{AD}| = |\overline{BD}|$, and $|\overline{AC}| < |\overline{AD}|$, $|\overline{AD}|^2 - |\overline{AC}|^2 = |\overline{CD}| = 25$. Find the value of $|\overline{DE}|$.



A) 16 B) 15 C) 14 D) 13 E) NOTA

5. For the equation $a^2 + b^2 + c^2 = 15^2$, where a, b, and c are integers, how many solutions exist?

A) 96 B) 144 C) 150 D) 192 E) NOTA

6. The graph of $(x^2 + y^2 - 2x + 4y + 5)(x^2 + y^2 + 4x + 2y + 5) = 0$ consists of two points. Find the distance between the two points.

A) $2\sqrt{2}$ B) $\sqrt{10}$ C) $\sqrt{13}$ D) 4 E) NOTA

7. A polynomial f has real coefficients. If $f(1) = -1$, $f(2) = 3$, $f(3) = 9$, $f(4) = 16$, and $f(5) = 24$, and if f has least possible degree, find the value of $f(7)$.

A) 48 B) 54 C) 60 D) 66 E) NOTA

8. Find the smallest positive angle θ such that $\left(\cos 37^\circ\right)\left(\cos 53^\circ\right)=\left(\sin 37^\circ\right)\left(\sin\theta\right)$.

A) 21° B) 37° C) 53° D) 69° E) NOTA

9. A right triangle with hypotenuse of length 6 has one acute angle of measure $\sin^{-1}\left(\frac{\sqrt{13}-\sqrt{5}}{6}\right)$. Find the length of the triangle's altitude to its hypotenuse.

A) 1 B) $\frac{2}{3}$ C) 2 D) $\frac{4}{3}$ E) NOTA

10. Find the sum of the values of x that satisfy the equation $\begin{vmatrix} 1 & 2 & 3 \\ x & 5 & 0 \\ 4 & x & 6 \end{vmatrix}=0$.

A) 3 B) 4 C) 5 D) 6 E) NOTA

11. For the function $f(x)=\frac{2x-1}{1+2x}$, find the nonzero value c such that $f\left(f(c)\right)=f(c-1)$.

A) 1 B) 1.5 C) 2 D) 2.5 E) NOTA

12. For the data set $\{5,1,3,7,3,7,12,5,2,7,7,1\}$, let A be the mean, B be the median, and C be the mode. Find the value of $C-A-B$.

A) -3 B) -1 C) 3 D) 1 E) NOTA

13. Find the area enclosed by a regular hexagon with apothem of length 4.

A) $8\sqrt{3}$ B) $16\sqrt{3}$ C) $24\sqrt{3}$ D) $32\sqrt{3}$ E) NOTA

14. Find the value of $-\sqrt{5-\sqrt{5-\sqrt{5-\sqrt{5-\ldots}}}}$.

A) $\frac{1+\sqrt{21}}{2}$ B) $\frac{-1+\sqrt{21}}{2}$ C) $\frac{1-\sqrt{21}}{2}$ D) $\frac{-1-\sqrt{21}}{2}$ E) NOTA

15. Sierpinski's triangle is formed by taking a solid black equilateral triangle and removing (changing to white) the middle fourth of it, then removing the middle fourths of the remaining triangles, then removing the middle fourths of the remaining triangles, ad infinitum. The first five iterations in the creation of Sierpinski's triangle are as follows:

What fraction of the area of the original solid triangle (seen first above) remains black after an infinite number of removals to form Sierpinski's triangle?

A) 0 B) $\frac{1}{16}$ C) $\frac{1}{8}$ D) $\frac{1}{4}$ E) NOTA

16. Find the value of $\prod_{n=1}^{18}\left(2^n \sin\left(20n-10\right)^\circ\right)$.

A) 2^{155} B) 2^{163} C) -2^{155} D) -2^{163} E) NOTA

17. What is the largest solution to the equation $\sin x = 0.5$, where $x \le \frac{241\pi}{12}$?

A) $\frac{109\pi}{6}$ B) $\frac{113\pi}{6}$ C) $\frac{115\pi}{6}$ D) $\frac{119\pi}{6}$ E) NOTA

18. What is the sum of the five least nonnegative even integers and the four greatest odd negative integers?

A) 4 B) 8 C) 10 D) 14 E) NOTA

19. Find the sum of the reciprocals of the positive integral factors of 28.

A) 2 B) 3 C) 4 D) 5 E) NOTA

20. A square is formed by connecting, in consecutive order, the midpoints of alternating sides of a regular octagon. Find the ratio of the area enclosed by the square to the area enclosed by the regular octagon.

A) $\frac{5}{8}$ B) $\frac{3+2\sqrt{2}}{8}$ C) $\frac{1+\sqrt{2}}{4}$ D) $\frac{4+\sqrt{2}}{8}$ E) NOTA

21. Simplify: $(\csc\theta-\cot\theta)(\csc\theta+\cot\theta)$

A) $\tan^2\theta$ B) $1+\sin\theta$ C) $\frac{1+\cos\theta}{2}$ D) $\frac{1+\sin\theta}{2}$ E) NOTA

22. Find the value of $\sum_{n=1}^{\infty}\left(\left(n^2+2n-4\right)\left(\frac{2}{3}\right)^n\right)$.

A) 31 B) 34 C) 37 D) 40 E) NOTA

23. Find the values of x that satisfy the equation $\cos\left(\frac{4\pi}{3}-\cos^{-1}x\right)=x$.

A) $\pm\frac{\sqrt{3}}{2}$ B) $\pm\frac{1}{2}$ C) $-\frac{\sqrt{3}}{2}$ D) $-\frac{1}{2}$ E) NOTA

24. Simplify: $(\sin x)(\sin(3x))+(\cos x)(\cos(3x))$

A) $\cos^2 x$ B) $\sin^2 x$ C) $\cos(2x)$ D) $\csc x$ E) NOTA

25. Find the length of the latus rectum of the conic section with equation $(x-4)^2-(y+3)^2=1$.

A) 2 B) $\sqrt{2}$ C) 3 D) $\sqrt{3}$ E) NOTA

26. Find the sum of all positive integers m that make $\frac{\left(1-\sqrt{3}i\right)^{24}}{\left(\sqrt{2}+\sqrt{2}i\right)^m}$ an integer. Note that $i=\sqrt{-1}$.

A) 48 B) 60 C) 72 D) 84 E) NOTA

27. In how many ways can Special Agent Dale Cooper walk to the 17th stair of the 17-stair staircase in the Black Lodge if he only goes up stairs two or three at a time? A "way" consists of a sequence of stair numbers that Special Agent Dale Cooper steps on. Special Agent Dale Cooper's first step from the floor can be either to the 2nd or 3rd stair.

A) 48 B) 49 C) 64 D) 65 E) NOTA

28. Find the sum of the cubes of the solutions to the equation $2x^4 - 3x^2 - 3x + 7 = 0$.

A) -1.5 B) 3 C) 4.5 D) 8 E) NOTA

29. Everett and Sydney plan to attend a party that lasts from 10 P.M. until 6 A.M. the next day. Sydney plans to stay at the party for four continuous hours, then leave (hence Sydney can't arrive at the party any later than 2 A.M.). Everett plans to stay at the party for six continuous hours, then leave (hence Everett can't arrive at the party any later than midnight). What is the probability that Everett and Sydney are at the party at the same time for at least three hours?

A) $\frac{11}{16}$ B) $\frac{5}{8}$ C) $\frac{13}{16}$ D) $\frac{7}{8}$ E) NOTA

30. A regular pentagon has sides each with length $\cos 36^\circ$. The five diagonals of this pentagon are drawn, and the five points of pairwise intersections of these diagonals form another regular pentagon. Find the length of a side of this new regular pentagon.

A) $\cos 54^\circ$ B) $\cos 72^\circ$ C) $\frac{\cos 18^\circ}{2}$ D) $\frac{\cos 36^\circ}{2}$ E) NOTA