Matrices & Vectors
Alpha, Round 2
Test #423

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. What is the value of the determinant of a 2x2 matrix whose entries on the main diagonal are the two largest two-digit triangular numbers and whose entries on the antidiagonal are the two smallest two-digit pentagonal numbers?

   (A) 6676 (B) 9291 (C) 6756 (D) 6834 (E) NOTA

2. If Kirby picks 4 elements at random from the collection {-2, 0, 1, 2, 0} and proceeds to randomly construct a 2x2 matrix using each of his selected elements exactly once, what is the probability that his resulting 2x2 matrix is invertible?

   (A) $\frac{2}{5}$ (B) $\frac{5}{12}$ (C) $\frac{1}{5}$ (D) $\frac{1}{10}$ (E) NOTA

3. If the reduced row echelon form of a matrix *A* is given by *B*, what is the value of *B* when $A = \begin{bmatrix} 5 & 66 & 30 \\ -17 & 81 & 23 \\ -9 & -21 & 69 \end{bmatrix}$?

   (A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

   (C) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (E) NOTA

4. Evaluate: $\begin{bmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{bmatrix}^5$

   (A) $\begin{bmatrix} -1024 & -1024\sqrt{3} \\ 1024\sqrt{3} & -1024 \end{bmatrix}$ (B) $\begin{bmatrix} -512 & 512\sqrt{3} \\ -512\sqrt{3} & -512 \end{bmatrix}$

   (C) $\begin{bmatrix} 512 & 512\sqrt{3} \\ -512\sqrt{3} & 512 \end{bmatrix}$ (D) $\begin{bmatrix} -1024 & -1024\sqrt{3} \\ -1024\sqrt{3} & -1024 \end{bmatrix}$ (E) NOTA

5. Which of the following matrices is/are nonsingular?

I. $\begin{bmatrix} 8 & 7 & 9 \\ 5 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix}$ II. $\begin{bmatrix} 27 & 3 \\ 54 & -12 \end{bmatrix}$ III. $[0]$ IV. $\begin{bmatrix} 4 & -4\sqrt{3} \\ 14\sqrt{3} & -42 \end{bmatrix}$

(A) I, III, IV only (B) II only

(C) I, III only (D) II, IV only (E) NOTA

6. Which of the following matrices is nilpotent?

I. $\begin{bmatrix} e & 1 \\ 0 & \pi \end{bmatrix}$ II. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ III. $\begin{bmatrix} 4 & -2\sqrt{3} \\ 16\sqrt{3} & -24 \end{bmatrix}$ IV. $\begin{bmatrix} 2 & -3 \\ \frac{4}{3} & -2 \end{bmatrix}$

(A) I (B) II (C) III (D) IV (E) NOTA

7. If A is a 6 X 6 matrix, and $|A| = 8$, what is $\det(4A)$?

(A) 32768 (B) 16384 (C) 65536 (D) 8192 (E) NOTA

8. Which of the following matrices is positive semidefinite?

I. $\begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix}$ II. $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ III. $\begin{bmatrix} -1 & -2\sqrt{6} \\ 8\sqrt{3} & -12 \end{bmatrix}$ IV. $\begin{bmatrix} 2 & -6 \\ -5 & -8 \end{bmatrix}$

(A) I (B) II (C) III (D) IV (E) NOTA

9. What is the adjoint of the matrix $\begin{bmatrix} 2 & 5 & 3 \\ 4 & 9 & -2 \\ 0 & 3 & 1 \end{bmatrix}$?

(A) $\begin{bmatrix} 15 & 4 & 37 \\ 4 & 2 & 16 \\ 12 & -6 & -2 \end{bmatrix}$ (B) $\begin{bmatrix} 15 & 4 & 12 \\ 4 & 2 & -6 \\ 37 & 16 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & -4 & 12 \\ 4 & 2 & -6 \\ -37 & 16 & -2 \end{bmatrix}$ (D) $\begin{bmatrix} 15 & 4 & -37 \\ -4 & 2 & 16 \\ 12 & -6 & -2 \end{bmatrix}$ (E) NOTA

10. Evaluate: $2\begin{bmatrix} 4 & 14 & 41 \\ -5 & 33 & -16 \\ -29 & 53 & 7 \end{bmatrix} + 3\begin{bmatrix} 6 & 9 & -2 \\ 2 & 0 & 20 \\ 5 & 7 & 10 \end{bmatrix}$

(A) $\begin{bmatrix} 26 & 55 & 76 \\ -4 & 66 & 28 \\ -43 & 127 & 44 \end{bmatrix}$ (B) $\begin{bmatrix} -24 & 60 & 119 \\ -11 & 99 & -8 \\ -77 & 173 & 41 \end{bmatrix}$

(C) $\begin{bmatrix} 26 & 55 & 88 \\ -16 & 69 & 31 \\ -43 & 127 & 44 \end{bmatrix}$ (D) $\begin{bmatrix} -24 & 60 & 127 \\ -19 & 101 & -6 \\ -77 & 173 & 41 \end{bmatrix}$ (E) NOTA

11. Joyce blindly picks three numbers out of a hat containing the integers -2, 4, 5, 3, 7, and 0. Nick then collects the remaining three numbers from the hat. If they each construct a vector using the integers drawn, what is the probability that the length of Joyce's vector is larger than that of Nick's?

(A) $\frac{9}{20}$ (B) $\frac{1}{2}$ (C) $\frac{11}{20}$ (D) $\frac{3}{5}$ (E) NOTA

12. In a Markov chain model, the steady state vector of a stochastic matrix is an eigenvector of the stochastic matrix with eigenvalue of ___ ?

(A) 1 (B) 0 (C) -1 (D) 0.5 (E) NOTA

13. Evaluate: $\begin{vmatrix} e & -4e & 3e \\ e+1 & 2e & 5 \\ -e & 1 & 0 \end{vmatrix}$

(A) - $6e^3 + 17e^2 - 8e$ (B) $3e^3 - 2e^2 + 3e$ (C) $20e^2 - 5e$

(D) $6e^3 + 23e^2 - 2e$ (E) NOTA

14. Consider the convex quadrilateral with vertices at (-6, 4), (3, 5), (4, -2), and (-5,2). If each of the vertices is shifted according to the matrix $\begin{bmatrix} -4 & 3 \\ -6 & -2 \end{bmatrix}$, what is the area of the resulting region?

(A) 804 (B) 1014 (C) 614 (D) 1084 (E) NOTA

15. Consider the 3-dimensional vectors U, V, and W and let × denote the cross product. Which of the following properties is/are true?

I. $U \times (V \times W) = (U \times V) \times W$
II. $If\ U \times V = U \times W,\ then\ V = W$
III. $U \times (V + W) = (U \times V) + (U \times W)$
IV. $(U \times V) \times (U \times W) = \big(U \cdot (V \times W)\big)U$

(A) I, III, IV only (B) III, IV only

(C) I, III only (D) II, III, IV only (E) NOTA

16. Given the points (2, 0, -3) and (-3, 5, 4), find the equation of the line that passes through these points. Express your answer in symmetric form.

(A) $\frac{x-2}{5} = \frac{y}{5} = \frac{z+3}{7}$ (B) $\frac{x}{5} = \frac{y}{-5} = \frac{z}{7}$

(C) $\frac{x-2}{5} = \frac{y}{-5} = \frac{z+3}{-7}$ (D) $\frac{x+2}{5} = \frac{y}{-5} = \frac{z-3}{7}$ (E) NOTA

17. What is the equation of the plane that contains the points (4, 7, 2), (0, -2, 5), (3, 5, 6)? Express your answer in standard form.

(A) $30x - 13y + z - 31 = 0$ (B) $10x - y + z - 35 = 0$

(C) $3x - 3y + z - 11 = 0$ (D) $2x + 3y + z - 21 = 0$ (E) NOTA

18. If $T = (2, 4, -5)$ and $Z = (6, 9, 17)$, what is the direction cosine of vector $\overrightarrow{TZ}$ along the z-axis?

(A) $\frac{22\sqrt{21}}{105}$ (B) $\frac{\sqrt{21}}{21}$ (C) $\frac{4\sqrt{21}}{105}$ (D) $\frac{5\sqrt{21}}{4}$ (E) NOTA

19. Given $\vec{u} = \langle -2,5,7 \rangle$ and $\vec{v} = \langle 3,1,4 \rangle$, evaluate $proj_{\vec{v}}\vec{u}$.

(A) $\frac{9}{26}\langle -2,5,7 \rangle$ (B) $\frac{9}{\sqrt{26}}\langle -2,5,7 \rangle$ (C) $\frac{27}{26}\langle 3,1,4 \rangle$ (D) $\frac{27}{\sqrt{26}}\langle 3,1,4 \rangle$ (E) NOTA

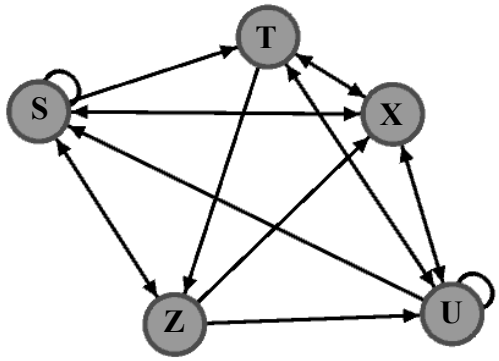
20. What is the row rank of A^T, given $A = \begin{bmatrix} 14 & -6 & -1 & 13 \\ 21 & -5 & 7 & -16 \\ 5 & 2 & 1 & 4 \end{bmatrix}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

21. What is the volume of the tetrahedron with vertices at the points (1, 4, 9), (-3, 0, 7), (5, -2, 2), and (-4, 7, 0)?

(A) 98 (B) 147 (C) 86 (D) 129 (E) NOTA

22. Given the following directed graph *G*, how many paths of length 3 are there from T to S?



(A) 4 (B) 8 (C) 3 (D) 5 (E) NOTA

23. Consider a game with payoff matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -2 & 1 & 2 \end{bmatrix}$, where A_{ij} denotes the number of points Taz receives from Sheila if Taz chooses option *i* and Sheila chooses option *j*. If Taz wishes to maximize his game score, what is his optimal strategy?

(A) $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ (B) $\begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$ (C) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ (E) NOTA

24. What is the shortest distance between the lines $x + 2 = \frac{y-3}{2} = \frac{z-5}{3}$ and $-x = \frac{y-1}{3} = \frac{z-4}{7}$?

(A) $\frac{\sqrt{6}}{6}$ (B) $\frac{5\sqrt{6}}{6}$ (C) 5 (D) $\sqrt{6}$ (E) NOTA

25. If *A* is a 5 x 7 matrix with rank 6 and *B* is a 7 x 9 matrix with rank 7, then which of the following cannot be the rank of *AB*?

(A) 5* (B) 6 (C) 7 (D) 8 (E) NOTA

26. The diagonalization of a matrix A is given by $A = PDP^{-1}$, where D is a diagonal matrix whose entries are the eigenvalues of A. In the diagonalization of $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix}$, what is the value of P? Assume that the eigenvalues are written such that $D_{kk} > D_{nn}$ for $k > n$.

(A) $\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (E) NOTA

27. In performing an LU decomposition of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 4 \\ 2 & 1 & -2 \end{bmatrix}$, Walter computed L, but forgot to write down the elements. Given $L = \begin{bmatrix} u & 0 & 0 \\ v & w & 0 \\ x & y & z \end{bmatrix}$, what is the value of $\frac{uvx}{wyz}$?

(A) $\frac{16}{3}$ (B) 12 (C) 3 (D) $\frac{11}{3}$ (E) NOTA

28. Evaluate: $\begin{vmatrix} 4 & 5 & -7 & 1 \\ 2 & 1 & 4 & 6 \\ 0 & 3 & 2 & -1 \\ -1 & 4 & 1 & 0 \end{vmatrix}$

(A) -137 (B) -459 (C) -418 (D)-295 (E) NOTA

29. Given $A = \begin{bmatrix} 4 & 7 & 0 \\ -2 & 3 & 1 \\ 2 & 5 & 2 \end{bmatrix}$ and $D^T = \begin{bmatrix} -2 & 7 & 2 \\ 1 & 9 & 0 \\ 3 & -5 & 2 \\ 11 & 4 & 6 \end{bmatrix}$, what is the value of $(AD)^T$?

(A) $\begin{bmatrix} 41 & 27 & 35 \\ 67 & 25 & 47 \\ -23 & -19 & -15 \\ 72 & -4 & 54 \end{bmatrix}$ (B) $\begin{bmatrix} -18 & 17 & 11 \\ -14 & 34 & 9 \\ 26 & 16 & -1 \\ 48 & 119 & 16 \end{bmatrix}$

(C) $\begin{bmatrix} -18 & 18 & 11 \\ -14 & 34 & 9 \\ 27 & 15 & 1 \\ 48 & 119 & 16 \end{bmatrix}$ (D) $\begin{bmatrix} 41 & 27 & 34 \\ 67 & 25 & 47 \\ -23 & 19 & -15 \\ 72 & -5 & 54 \end{bmatrix}$ (E) NOTA

30. What is the sum of the algebraic and geometric multiplicities of the real eigenvalues of $\begin{bmatrix} 3 & 6 & -1 \\ -7 & -9 & 3 \\ 1 & 6 & 0 \end{bmatrix}$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA