2003 Mu Alpha Theta National Convention
Complex Numbers Topic Test – Alpha Division

Notes for this test: Given a complex number z, the real and imaginary parts of z are given by $\text{Re}(z)$ and $\text{Im}(z)$, respectively. In addition, assume all arguments of trigonometric functions are in radians unless accompanied by the degree ($^\circ$) symbol.

1. Which of the following numbers is/are purely imaginary?

   I. $\sqrt{2}+\sqrt{-\pi}$

   II. $-|3+4i|\bar{i}$

   III. $711^{82}i^3+1110^{82}i^2$

   IV. $\sqrt{4-x^2}$, where x is a real number greater than 2.

   (A) I and IV only (B) II and III only

   (C) II and IV only (D) I and III only (E) NOTA

2. Given real numbers a and b such that $27-64i=a^3-b^3i$, find $a+b$.

   (A) 11 (B) 7 (C) 1 (D) -1 (E) NOTA

3. Evaluate: $25+\dfrac{3i+2}{6}-\dfrac{i}{2}$

   (A) $26+i$ (B) $\dfrac{i+76}{3}$ (C) $\dfrac{76}{3}$ (D) 27 (E) NOTA

4. Evaluate: $(3\sqrt{2}-i)(5i\sqrt{2}-2)$

   (A) $8\sqrt{2}+28i\sqrt{2}$ (B) $-8\sqrt{2}+28i$

   (C) $32i-\sqrt{2}$ (D) $28i-11\sqrt{2}$ (E) NOTA

5. Write $\dfrac{4+i}{2-3i}$ in the form $a+bi$, where a and b are real.

   (A) $\dfrac{5}{13}+\dfrac{14}{13}i$ (B) $1+2i$ (C) $\dfrac{5}{2}-\dfrac{3}{2}i$ (D) $\dfrac{2}{17}-\dfrac{5}{17}i$ (E) NOTA

6. Evaluate $y!+\sin\dfrac{\pi}{x}$ if $(2-i)x+(5i+1)y=7+13i$, and $x,y\in\mathbb{R}$.

   (A) $\dfrac{19}{2}$ (B) $2+\dfrac{\sqrt{3}}{2}$ (C) 5 (D) 7 (E) NOTA

7. What is the sum of the non-real roots of $f(x)=5x^3-3x^2-2$?

   (A) 1 (B) $\dfrac{3}{5}$ (C) $-\dfrac{2}{5}$ (D) -1 (E) NOTA

8. Find $r + 1/r$ if r is a root of $f(x) = x^2 - 13x + 1$.

   (A) -11 (B) 13 (C) $\sqrt{165}$ (D) 1 (E) NOTA

9. Let z be a complex number. What is the area of the graph of the region $|z - 2 - 3i| \leq 4$ in the complex plane?

   (A) 2π (B) 16π (C) 4π (D) 8π (E) NOTA

10. Which of the following is a fourth root of $16i$?

    (A) $2e^{9\pi i/8}$ (B) $2e^{7\pi i/8}$ (C) $2e^{3\pi i/8}$ (D) $2e^{\pi i/4}$ (E) NOTA

11. Let S be the set of all nonreal complex numbers of absolute value 1. If an element z is randomly chosen from S, what is the probability $(z-1)(z+1)^{-1}$ is a pure imaginary number?

    (A) 0 (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) NOTA

12. Which of the following matrices is Hermitian?

    (A) $\begin{pmatrix} 3 & 2-i & 1-2i \\ 2+i & -1 & -3i \\ 1+2i & 3i & 4 \end{pmatrix}$ (B) $\begin{pmatrix} -2i & 5+i & -1-3i \\ -5+i & i & 6 \\ 1-3i & -6 & 3i \end{pmatrix}$

    (C) $\begin{pmatrix} 2+3i & 6i & 1+i \\ -6i & 4 & 8-3i \\ 1-i & -8+3i & 5i \end{pmatrix}$ (D) $\begin{pmatrix} 4+7i & 2+i & 7+7i \\ 1-3i & 1-i & -3-i \\ 5+4i & 1-2i & 7-5i \end{pmatrix}$

    (E) NOTA

13. Given the equation $(-2 - 2i\sqrt{3})e^{i\theta} = 4e^{1802\pi i/3}$, ($\theta$ being a real number) find the value of θ such that $|\theta|$ is as small as possible.

    (A) $\frac{5\pi}{3}$ (B) $\frac{10\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$ (E) NOTA

14. How many of the fifty-seven roots of unity are in the third quadrant of the complex plane?

    (A) 17 (B) 16 (C) 15 (D) 14 (E) NOTA

15. In the complex plane, what is the area of the triangle whose vertices are $2 + 3i$, $4 - 5i$, and $1 + i$?

    (A) $\frac{15}{2}$ (B) 6 (C) 5 (D) $\frac{5}{2}$ (E) NOTA

16. If $x = e^{73\pi i/180}$ and $y = e^{17\pi i/180}$, find the value of $\mathrm{Im}(x)\mathrm{Re}(y) + \mathrm{Re}(x)\mathrm{Im}(y)$.

    (A) $\frac{1}{2}$ (B) 0 (C) $\frac{\sqrt{3}}{2}$ (D) 1 (E) NOTA

17. Given $n = 3^{909}$, find i^n.

(A) i (B) -1 (C) $-i$ (D) 1 (E) NOTA

18. For a complex number z, let $f(z) = \overline{z}$. Evaluate

$$\sum_{n=0}^{2003} \left(f^{(3n)}(1+i) - f^{(3n+1)}(1+i) \right)$$

where $f^{(n)}(x)$ is the result when $f(x)$ is composed with itself n times. For example, $f^{(0)}(x) = x$, $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = f(f(x))$, and so on.

(A) 2 (B) 0 (C) $2i$ (D) -2 (E) NOTA

19. Let $p(z) = 2z^4 + az^3 + bz^2 + cz + 3$, where a, b, and c are real numbers. Find a such that $p(2) = p(i) = 0$.

(A) $\frac{3}{4}$ (B) $\frac{7}{3}$ (C) $-\frac{11}{2}$ (D) 5 (E) NOTA

20. If $z^3 = 1$ and $z \neq 1$, then evaluate $(1 - z + z^2)(1 + z - z^2)$.

(A) 4 (B) 3 (C) 2 (D) 1 (E) NOTA

21. If z is a complex number, then how many of the following is/are always true?

I. z is imaginary.

II. $|\sin z| \leq 1$

III. $z = |z| \left(\cos \operatorname{Arctan} \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} + i \sin \operatorname{Arctan} \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right)$

IV. If $\left(\frac{z - \operatorname{Re}(z)}{\operatorname{Im}(z)} \right)^a = \left(\frac{z - \operatorname{Re}(z)}{\operatorname{Im}(z)} \right)^b$, where $\operatorname{Im}(z) \neq 0$, then $a = b$.

(A) 1 (B) 0 (C) 2 (D) 3 (E) NOTA

22. What is the imaginary part of $(\cos 4 + i \sin 4 + 1)^{2003}$?

(A) $2^{2003} \cos^{2003} 2 \sin 4006$ (B) $\sin 8012$

(C) 0 (D) $2 \sin^{2003} 2 \cos 4006$ (E) NOTA

23. Let t be a real number and define

$$f(t) = \frac{t + 2i}{t - 2i} + \frac{t - 2i}{t + 2i}$$

Find the maximum value of $|f(t)|$.

(A) 2 (B) 4 (C) $\sqrt{2}$ (D) $2\sqrt{2}$ (E) NOTA

24. Suppose r, s, and t are the roots of $f(x) = x^3 + 2x^2 - 7x + 2$. Find the value of

$$\frac{r}{r-1} + \frac{s}{s-1} + \frac{t}{t-1}$$

(A) 5 (B) $\frac{13}{2}$ (C) 3 (D) $\frac{11}{2}$ (E) NOTA

25. Evaluate: $\prod_{n=1}^{40} \left(\cos\left(\frac{n+38}{39}\right)^\circ + i \sin\left(\frac{n+38}{39}\right)^\circ \right)$

(A) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (B) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(C) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (D) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ (E) NOTA

26. Consider the function f on problem 18. Which of the following is a property of f for all complex numbers x and y?

I. $f(x \pm y) = f(x) \pm f(y)$ II. $f(xy) = f(x)f(y)$ III. $\mathrm{Im}(f(x)) = \mathrm{Im}(x)$

(A) I and II only (B) I, and III only

(C) I only (D) I, II, and III (E) NOTA

27. Given that $x + 1/x = \sqrt{3}$, what is the greatest integer less than $x^{6000} + 1/x^{6000}$?

(A) 0 (B) -1 (C) 2 (D) 1 (E) NOTA

28. Three complex numbers a, b, and c add up to zero. Given that $ab + bc + ac = 1$, what is the value of $a^2 + b^2 + c^2$?

(A) -4 (B) 3 (C) -2 (D) 1 (E) NOTA

29. Suppose the solutions to $ix^2 + (1-5i)x = 1 - 8i$ are $x_1 = a + bi$ and $x_2 = c + di$, where a, b, c, and d are all real and $a > c$. Find the inverse of the matrix given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(A) $\frac{1}{7}\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$ (B) $\frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$

(C) $\frac{1}{7}\begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix}$ (D) Inverse does not exist. (E) NOTA

30. For all real values m and n, let $f(m) = (m-2) + (3m-4)i$ and $g(n) = (-5n+6) + (-7n+8)i$. If $f(m_0) = g(n_0) = c$, what is the value of $(m_0 + n_0)c$?

(A) $-3 - 5i$ (B) $36 + 46i$ (C) $1 - 7i$ (D) $-5 + 11i$ (E) NOTA