Mu Alpha Theta National Convention 2003
Alpha Gemini Topic Test Solutions

The correct answer to each question is given immediately after the question number in parentheses.

1. (B) 1-9, there are 5. 10-99, there are 5 times 5=25. 100-999, there are 5 to the third power=125. 1000-9999, there are 5 to the fourth power=625. The sum is 780.

2. (B) You must have more than one side length to determine an area…otherwise the size of the object can change with all of the unknowns legs varying in size proportionately. If you have two side lengths, the figure becomes fixed in size, and you can determine the area based on the side lengths you know and the angles (this problem is especially easy if you know the lengths of the two sides with the 90˚ angle between them).

3. (D) $.\overline{567}_8 = 5/8 + 6/64 + 7/512 + 6/4096 + 7/32768 + \cdots = 5/8 + (6/64)/[1-(1/64)] + (7/512)/[1-(1/64)] = 185/252$

4. (D) $(7x-3y)/(2x+3y) = 6 \to x = -21y/5 \to (x-y)/(x+y) = [-(21y/5)-y]/[-(21y/5)+y] = 13/8$.

5. (A) 2520 distinct permutations exist that start with the letters AA, so the 2003rd one must start with AA. 360 permutations start with AAI; another 360 with AAL; 720 with AAN; 360 with AAO; and 360 with AAS, which brings the total to 2160 (so our answer should start with AAS). Permutations 1801-1860 start with AASI; 1861-1920 with AASL; 1921-2040 with AASN. Permutations 1921-1944 start with AASNI; 1945-1968 with AASNL; 1969-1992 with AASNN; and 1993-2016 with AASNO. Permutations 1993-1998 start with AASNOI; 1999-2004 with AASNOL. You can deduce that the 2003rd is AASNOLTIN.

6. (C) $\sqrt{ab} = 14$, and $2ab/(a+b) = 12 \to (a+b)/2 = ab/12 = 196/12 = 49/3$.

7. (C) Call the equal legs length x, and half of the base length y. Then $2x + 2y = 36 \to x + y = 18$. The altitude to the base lets us know that $9^2 + y^2 = x^2 \to x^2 - y^2 = 81 \to (x-y)(x+y) = 81 \to x - y = 4.5$. Solving for x and y gets that $x = 45/4$ and $y = 27/4$. Now use the altitude again to solve for the altitude to one of the legs. $Area = (1/2)(27/2)9 = (1/2)(45/4)Alt. \to Alt. = 54/5$.

8. (D) Only one way exists for Alex to get all five questions right, and the probability of that is $(3/5)^5 = 243/3125$. There are five ways he can miss only one question, $5(3/5)^4(2/5) = 810/3125$. There are 10 ways he can miss exactly two questions, $10(3/5)^3(2/5)^2 = 1080/3125$. Adding this up…2133/3125.

9. (B) $1/BY = (1/AX) + (1/CZ) \to BY = 20/9$.

10. (A) 2003 divided by 7 is 286.*. 2003 divided by 49 is 40.*. 2003 divided by 343 is 5.*. So there are 331 factors of 7 in 2003!, and the largest possible n is 331.

11. (D) The roots are $\pi/4,\ 2\pi/3,\ 3\pi/4,\ 5\pi/4,\ 4\pi/3$, and $7\pi/4$, so the sum is 6π.

12. (D) The length of the median to the z side of an x, y, z triangle is: $\sqrt{2x^2+2y^2-z^2}/2$. The largest value would be for $x = 8,\ y = 9$, and $z = 7$, which would be $\sqrt{241}/2$.

13. (B) $a+b+c = -4$, $ab+ac+bc = 7$, and $abc = -2$. The sum of the new roots will be $1/a + 2 + 1/b + 2 + 1/c + 2$, which is $(bc+ac+ab+6abc)/(abc) = (7-12)/-2 = 5/2$. Similarly, the sum of the product of two roots will be $[a+b+c+4(ab+ac+bc)+12abc]/(abc) = [-4+4(7)+12(-2)]/-2 = 0$. Finally, the product of all the roots will be $[1+2(a+b+c)+4(ab+ac+bc)+8abc]/(abc) = [1+2(-4)+4(7)+8(-2)]/-2 = -5/2$. So a new equation could be: $2x^3 - 5x^2 + 5 = 0$.

14. (B) $39 = T_S + (46-T_S)e^{10k} \to k = (1/10)\ln[(39-T_S)/(46-T_S)]$. $33 = T_S + (46-T_S)e^{2\ln[(39-T_S)/(46-T_S)]} \to T_S = -3^\circ\text{C}$.

15. (C) $P_t = P_0[1+(r/k)]^{kt}$. For I, $r = .08,\ k = 2 \to P_2 = \$5849.30$; for II, $r = .07,\ k = 4 \to P_2 = \$5859.29$; for III, $r = .06,\ k = 365 \to P_2 = \$5862.95$. For IV, $P_t = P_0e^{rt}$. For $r = .05 \to P_2 = \$5857.40$. From largest to smallest, III, II, IV, I.

16. (E) $y = \dfrac{x^5+5x^4-9x^3-49x^2-8x+60}{x^4-5x^3-7x^2+41x-30} = \dfrac{(x+5)(x+2)^2(x-1)(x-3)}{(x+3)(x-1)(x-2)(x-5)}$. So the asymptotes parallel to the x-axis are $x = -3,\ 2$, and 5. For $x = 1$, you have a hole. To find the oblique asymptote, you need to divide the numerator by the denominator. This results in $y = x + 10 + (48x^3 - 20x^2 - 548x + 360)/(x^4 - 5x^3 - 7x^2 + 41x - 30)$, so as x goes to infinity, y goes to $x + 10$.

17. (B) $\begin{vmatrix} x & 7 & 0 \\ 2 & x-6 & 7 \\ 0 & -1 & x \end{vmatrix} = 0 \to x^3 - 6x^2 - 7x = 0 \to x(x+1)(x-7) = 0 \to x = -1, 0,$ and 7.

18. (D) Use Kramer's Rule to solve for a, b, and c. This gets an ordered triple of $(6,-12,18)$. So the sum is 12.

19. (A) For a question like this, if n is the number of variables and r is the exponent, then the answer is of the form: $(r+1)(r+2)\cdots(n+r-1)/(n-1)$. In this case n is 4 and r is 5. $(6\cdot 7\cdot 8)/3! = 56$.

20. (E) Each zero comes from a multiplication of 2 and 5 in the factorial product. Since there are plenty of twos, we should look for the number of fives. To find the number of fives in $n!$, divide n by 5, 25, 125, etc…and count up the result. The number 8020 divided by 5 is 1604; divided by 25 is 320 (truncated); divided by 125 is 64; divided by 625 is 12; and divided by 3125 is 2. This adds up to 2002. The next factorial that will add a factor of 5 is 8025…but that will add two factors of five to bring the total up to 2004. So there are no natural numbers n such that $n!$ ends in exactly 2003 zeroes.

21. (C) If you draw the triangle out, you see that it is a 6-8-10 triangle with the shortest side parallel to the y-axis. If we call the angle opposite this side A, then $\cos A = 4/5$. By the half angle formula, $\cos(A/2) = 3\sqrt{10}/10$. We can use this to solve for the length of the angle bisector, a (the hypotenuse of a new right triangle). This length is $3\sqrt{10}/10 = 8/a \rightarrow a = 8\sqrt{10}/3$. We can also solve for where the angle bisector intersects the left side of the triangle $b+4\ldots\left(8\sqrt{10}/3\right)^2 - 8^2 = 64/9 \rightarrow b = 8/3$. So two points on a line with the angle bisector are (10,4) and (2,20/3). The equation for this line is $x + 3y = 22$, and the y-intercept is 22/3.

22. (C) I solved this problem graphically. I put one side of a triangle on the x-axis from the point (0,0) to the point (A,0). I then have another point out at (B,C). The median to the side with the x-axis goes to the point (A/2,0). The median to the side between (A,0) and (B,C) is [(A+B)/2,C/2]. The equations for the lines containing these two medians are: $y = Cx/(A+B)$ and $y = (2Cx - AC)/(2B - A)$. They intersect at the point [(A+B)/3,C/3], which divides the median into a ratio of 2:1.

23. (A) The vector which is perpendicular to the plane is $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. A line through the point (1,1,3) and parallel to this vector is given by $x = 1 + 3t$, $y = 1 + 2t$, and $z = 3 + 6t$. This line intersects the plane at one point, and we can plug in x, y, and z to solve for t. $3(1+3t) + 2(1+2t) + 6(3+6t) = 6 \rightarrow t = -17/49$. Plugging this t into our x, y, and z equations gives $(-2/49, 15/49, 45/49)$.

24. (D) To solve this question I set up parametric equations for the location of each planet. $x_e = \cos(2\pi t/365)$, $y_e = \sin(2\pi t/365)$, $x_m = A\cos(2\pi t/687)$, and $y_m = A\sin(2\pi t/687)$. In this coordinate system, the sun is at the origin, the earth's radius is 1, and Mars's radius is A. At time 0 in this system, the planets are closest together. I need to find the next time t when the planets are $A-1$ apart. $(x_m - x_e)^2 + (y_m - y_e)^2 = (A-1)^2 \rightarrow [A\cos(2\pi t/687) - \cos(2\pi t/365)]^2 + [A\sin(2\pi t/687) - \sin(2\pi t/365)]^2 = (A-1)^2 \rightarrow \cos[(2\pi t/687) - (2\pi t/365)] = 1$. The next time the planets are closest together is when $t = 250755/322 \approx 779$ days.

25. (C) The discriminant $B^2 - 4AC$ is invariant through any rotation of the coordinate axes. If we rotate to get an equation without the xy term, then we'll have that $B^2 - 4AC = -4A'C'$, where the new equation will be $A'x^2 + C'y^2 = 1$. The area of this ellipse is $\pi/\sqrt{A'C'} = 2\pi/\sqrt{4AC - B^2} = \pi/2$.

26. (E) 26 (ABC, ABD, ABE, ABG, ABI, ACD, ADE, AEG, AEI, BCE, BDE, BDG, BDI, BEI, BFG, BFH, BGI, BHI, CDE, DEG, DEI, DFG, EGI, EHI, FGH, and GHI).

27. (D) $\vec{\mathbf{A}} \bullet \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\cos\theta \rightarrow (1)(3) + (2)(-4) + (-5)(-5) = 20 = \sqrt{(1)^2 + (2)^2 + (-5)^2}\sqrt{(3)^2 + (-4)^2 + (-5)^2}\cos\theta \rightarrow$ $\cos\theta = 2\sqrt{15}/15 \rightarrow \sin\theta = \sqrt{165}/15$.

28. (B) $\sum_{n=1}^{\infty} \frac{2n}{6^n} = \frac{2}{6} + \frac{4}{36} + \frac{6}{216} + \cdots = \left(\frac{2}{6} + \frac{2}{36} + \frac{2}{216} + \cdots\right) + \left(\frac{2}{36} + \frac{2}{216} + \cdots\right) + \left(\frac{2}{216} + \cdots\right) + \cdots = \frac{(2/6)}{1-(1/6)} + \frac{(2/36)}{1-(1/6)} + \cdots = \frac{2}{5} + \frac{2}{30} + \cdots = \frac{(2/5)}{1-(1/6)} = \frac{12}{25}$.

29. (D) The rth term in the expansion of $(a+b)^n$ is $\binom{n}{r-1}a^{n-r+1}b^{r-1}$. In our case the third term will be without an x. So the third term will be: $\binom{6}{2}(2x)^4\left(\frac{3}{x^2}\right)^2 = \frac{6!}{2!\,4!}(16x^4)\left(\frac{9}{x^4}\right) = 2160$.

30. (B) $8x^2 - 32x - y + 36 = 0 \rightarrow y - 4 = 8(x-2)^2$. The focal width does not change with translation, so I change the equation I'm working with to: $y = 8x^2$. The focus of this parabola is at (0,1/32). Plugging this y in solves for the x locations at the end point of the line segment. $x = \pm 1/16$. So the focal width is 1/8.