Mu Alpha Theta National Convention 2003
Gemini for Calculus

For all questions, answer E. "NOTA" means none of the above answers is correct.

1. What is the 2003 rd derivative of f(x) = sin x ?

A. - sin x B.- cos x C. sin x D. cos x E. NOTA

2. Consider the solid such that the base is a circle with equation $x^2 + y^2 = 1$ and each cross section perpendicular to the base is an equilateral triangle. What is the volume, in cubic units, of the solid?

A. $\frac{\sqrt{3}}{3}$ B. $\frac{2\sqrt{3}}{3}$ C. $\sqrt{3}$ D. $\frac{4\sqrt{3}}{3}$ E. NOTA

3. Suppose a cylinder is to be inscribed in a right circular cone with radius R and height H. If the cylinder is to have maximum volume, then the radius of the cylinder should be

A. $\frac{R}{4}$ B. $\frac{R}{3}$ C. $\frac{R}{2}$ D. $\frac{2R}{3}$ E. NOTA

4. What is the equation of the line normal to the graph of $y = 5 - x^4$ at the point (1, 4)?

A. 4y-x=15 B. 4y+x=15 C. 4y-x=17 D. 4y+x=17 E. NOTA

5. The graph of $f(x) = \frac{x^5}{20} - \frac{x^4}{3} - \frac{4x^3}{3} + 16x^2 + 10x + 48$ has exactly k point(s) of inflection where k =

A. 1 B. 2 C. 3 D. 4 E. NOTA

6. A particle moves along a straight line with velocity, $v(t) = t^3 - 9t^2 + 23t - 15$. Through how many units does the particle move for $0 \le t \le 6$?

A. 0 B. 6.25 C. 10.25 D. 20.5 E. NOTA

7. Determine the area, in square units, of the region bounded by the graphs of $y = 4$, $y = x^2$, and $xy = 1$.

A. 6 - ln 4 B. $\frac{14 - 3\ln 4}{3}$ C. $\frac{14 - \ln 4}{3}$ D. 4 + ln 6 E. NOTA

8. If $f(x) = |1 - \ln x|$, then $f'(x)$ is greater than zero for

A. $0 < x < e$ B. $0 < x \leq e$ C. $x > e$ D. $x \geq e$ E. NOTA

9. $y = t^2 + 3t$ and $x = t^4 + t$. Determine the value of $\frac{d^2y}{dx^2}$ for $t = 1$.

A. $\frac{-2}{125}$ B. $\frac{-2}{25}$ C. $\frac{-2}{5}$ D. -2 E. NOTA

10. $\lim_{n \to \infty} \left(\frac{n^2 + an + b}{n^2 + cn + d}\right)^n =$ [a, b, c, and d are real numbers.]

A. 1 B. e^{a-c} C. e^{a+c} D. Infinity E. NOTA

11. Consider the region bounded by $y = \sin x$, $y = 0.5$, $x = \frac{\pi}{6}$, and $x = \frac{5\pi}{6}$. What is the volume, in cubic units, of the solid formed when the region is rotated around the X-axis? Round your answer to the nearest hundredth.

A. 2.34 B. 3.01 C. 4.65 D. 11.40 E. NOTA

12. Quiz question: $\int \sec^2 x \tan x \, dx =$ ___?___. Sue wrote $\frac{\tan^2 x}{2} + C$. Meg wrote $\frac{\sec^2 x}{2} + C$. Which girl answered correctly?

A. Sue not Meg B. Meg not Sue. C. Both D. Neither E.NOTA

13. For $a > 0$, $\int_a^{2a} \frac{x^2 + x - 2}{x^2} \, dx =$

A. $\ln 2$ B. $a + \ln 2 - \frac{1}{a}$ C. $a + \ln 2$ D. $a + \ln 2 + \frac{1}{a}$ E. NOTA

14. If four equal subdivisions are used, what is the value of the trapezoidal rule when it is applied to find an estimate for the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$?

A. $\frac{28}{17}$ B. $\frac{52}{31}$ C. $\frac{91}{57}$ D. $\frac{5}{3}$ E. NOTA

15. The area, in square units, of the larger region bounded by the graphs of $y = 2x^3 - 9x^2 + 11x - 1$ and $y = x - 1$ is

A. 2 B. 4 C. 6 D. 8 E. NOTA

16. What is the value of $c \in (3,8)$ which satisfies the mean value theorem for $f(x) = \sqrt{x+1}$?.

A. 5.25 B. 5.3 C. 5.5 D. 5.75 E. NOTA

17. The graphs of $x^2 + xy + y^2 = 7$ and $y = 2x$ have a point of intersection in the first quadrant. What is the measure of the acute angle of intersection at this point, to the nearest degree?

A. 78 B. 80 C. 82 D. 85 E. NOTA

18. Which of the given choices about the graph of a third degree polynomial is false?

A. There is at least one X intercept B. There is exactly one point of inflection
C. There exists at least one relative maximum
D. There is always a region over which the graph is concave down. E. NOTA

19. $\int \tan^2 4x \, dx =$

A. $\frac{\tan 4x}{4} - x + C$ B. $x - \frac{\tan 4x}{4} + C$ C. $\tan 4x + C$ D. $4 \tan x + C$ E. NOTA

20. What is the nth derivative of $f(x) = \frac{1}{x}$?

A. $\frac{n!}{x^{n+1}}$ B. $\frac{(-1)^n n!}{x^n}$ C. $\frac{(-1)^{n-1} n!}{x^{n-1}}$ D. $\frac{(-1)^n n!}{x^{n+1}}$ E. NOTA

21. For the graph of the parabola $ky = x^2$, where k is an integer, the area bounded by the parabola and its latus rectum is an integer if and only if k is a multiple of

A. 3 B. 4 C. 5 D. 6 E. NOTA

22. An old buried cylinder contains 20 gallons of water. At time t=0 water starts being pumped into the tank at the rate of 6 gallons per minute. At the same time a rust spot in the bottom causes a leak and the water leaks out at a rate of $\sqrt{t+6}$ gallons per minute. Assuming that this process continues for an hour and the tank never overflows, what is the maximum amount of water that will ever be in the tank? Select your answer correct to the nearest integer.

A. 66 gal. B. 86 gal. C. 96 gal. D. 106 gal E. NOTA

23. Suppose a substance decays at a rate equal to 1/10 the amount of the substance. What is the half-life, to the nearest year, of this substance?

A. 7 B. 8 C. 9 D. 10 E. NOTA

24. For $b > 0$, $\int_0^2 b^x \, dx =$

A. $\frac{b^3}{3}$ B. $\frac{b^3 - b}{3}$ C. $\frac{b^2 - 1}{\ln b}$ D. $\frac{b^2}{\ln b}$ E. NOTA

25. $\int_0^{2\pi} \sqrt{2a^2 - 2a^2 \cos\vartheta} \, d\vartheta =$

A. $2|a|$ B. $4|a|$ C. $8|a|$ D. $10|a|$ E. NOTA

26. The average value of the function $f(x) = \sqrt{3x+1}$ over the domain $0 \le x \le 5$ is

A. 2.6 B. 2.8 C. 2.9 D. 3.0 E. NOTA

27. How many real numbers c satisfy the equation $\int_c^{c^2} 3x^2 dx = \int_c^{c^2} 2x^3 dx$?

A. 2 B. 4 C. 6 D. 8 E. NOTA

28. Matt needs to fence a rectangular 1200 square foot region. The fencing costs \$3.00 per linear foot and his neighbor will pay half of the cost for the fence along the property line. If the cost for Matt is to be a minimum, what is the neighbor's share?

A. \$30.00 B. \$45.00 C. \$60.00 D. \$75.00 E. NOTA

29. The graph of $f(x) = 4x^3 - 6x^2 - 9x + k$ has three distinct X intercepts under the condition

A $-2.5 < k < 13.5$ B. $-3 < k < 10$ C. $0 < k < 15$ D. $2 < k < 14$ E. NOTA

30. A vertical pole 25 feet tall casts a shadow on a level plane. In the morning, the angle of elevation from the tip of the shadow to the sun is increasing at a rate of 0.0052 radians per minute. What is the rate of change, to the nearest hundredth, of the length of the shadow when the angle of elevation is 30°?

A. -0.52 ft/min B. -0.54 ft/min C. -0.58 ft/min D. -0.60 ft/min E. NOTA

GEMINI FOR CALCULUS 2003
SOLUTIONS

1. **B.** 2003/4 --> remainder 3

2. **D.** Area of one triangle: $\sqrt{3}(1-x^2)$

$$\text{Volume} = 2\sqrt{3}\int_0^1 (1-x^2)dx = 4\sqrt{3}/3$$

3. **D.** $\frac{H}{R} = \frac{h}{R-h}$; $h = \frac{H(R-r)}{R}$

$$V = \pi r^2\left(\frac{HR - Hr}{R}\right)$$

dV/dr = 0 for r = 2R/3

4. **A.** Slope of the normal at (1,4) is 1/4

(y - 4)/(x-1) = 1/4

4y - x = 15

5. **C.** $f''(x) = (x - \sqrt{8})(x + \sqrt{8})(x - 4)$

Sign changes at three points.

6. **D.** v(t) = (t - 1)(t - 3)(t-5)

$$s(t) = \frac{t^4}{4} - 3t^3 + \frac{23t}{2} - 15t$$

s(0) = 0; s(1) = -6.25; s(3) = -2.25;
s(5) = -6.25; s(6) = 0
Distance = 6.25 + 4 + 4 + 6.25 = 20.5

7. **B.** $\int_{.25}^{1}(4-1/x)dx - \int_1^2(4-x^2)dx$

8. **C.** f(x) = 1 - lnx for 0<x≤e;

f(x) = lnx - 1 for x >e
f '(x) = -1/x for 0<x<e; f '(x) = 1/x for x >e.
Abrupt change at x = e.

9. **C.** $dy/dt = 2t + 3$; $dx/dt = 4t^3 + 1$

$dy/dx = (2t+3)/(4t^3+1)$

$$\frac{d^2y}{dx^2} = \frac{2(4t^2+1) - 12t^2(2t+3)}{(4t^3+1)^2}\cdot\frac{dt}{dx}$$

For t = 1 this value is -2/5

10. **B.**

$$\text{change to } \left(1 + \frac{(a-c) + (b-d)/n}{n + c/n + d/n^2}\right)^n$$

as n tends to infinity this tends to e^{a-c}

11. **B.** $\pi\int_{\pi/6}^{5\pi/6}(\sin^2 x - 0.25)dx = 3.01$

12. **C.** Sue used u = tan x, $du = \sec^2 x\,dx$.

Meg used u = sec x, du = sec x tan x dx
Note: $\tan^2 x = 1 - \sec^2 x$. Here is the importance of C.

13. **B.** $\int_a^{2a}(1 + 1/x - 2x^{-2}dx =$

$$2a + \ln 2a + 1/a - a - \ln a - 2/a = a + \ln 2 - 1/a$$

14. **E.** 1/2[f(1)+2f(2)+2f(3)+2f(4)+f(5)]=

1/2[1 + 1+ 2/3 + 1/2 + 1/5] = 101/60

15. **B.** $\int_0^2 ([(2x^3 - 9x^2 + 11x - 1) - (x - 1)]dx$

$\int_0^2 (2x^3 - 9x^2 + 10x)dx = 4$

16. **A.** f(3) = 2 and f(8) = 3

$f'(x) = \frac{1}{2\sqrt{x+1}} = \frac{1}{5}$

$x = 5.25$

---

17. **A.** Point of intersection is (1,2)

For equation 1: dy/dx = (-2x-y)/(x + 2y)

at (1,2) slope is -4/5

For equation 2: slope is 2

Arctan $\theta = \left|\frac{2 + 4/5}{1 - 8/5}\right|$; $\theta = 78°$

18. **C.** Example: $y = x^3$

19. **A.** $\int (\sec^2 4x - 1)dx =$

(tan 4x)/4 - x + C

20. **D.** $f'(x) = -1/x$; $f''(x) = 2/x^3$;

$f'''(x) = -6/x^4$ . . .

21. **D.** $2\frac{k^2}{8} - 2\int_0^{k/2} \frac{x^2}{k} dx = \frac{k^2}{6}$

22. **A.** max occurs when input = output.

$6 = \sqrt{t+6}$; $t = 30$

Water = $20 + 180 - \int_0^{30} (t+6)^{1/2} dt = 66$

---

23. **A.** $\frac{1}{2} = e^{-t/10}$; $-10 \ln 0.5 = t$

24. **C.** $\int_0^2 e^{x \ln b} dx = \frac{e^{x \ln b}}{\ln b}$ from 0 to 2

$= \frac{b^2 - 1}{\ln b}$

25. **C.** Factor out |a|. Use calculator.

26. **B.** $\frac{\int_0^5 \sqrt{3x+1}\, dx}{5} = 2.8$

27. **B.** $c^6 - c^3 = c^8 - c^4$

$c^3(c-1)(c^4 + c^3 - 1) =$

c=0, c=1, the fourth degree equation

has two real roots.

28. **C.** Cost = $6x + \frac{3600}{x} + \frac{1800}{x}$

$C' = 6 - \frac{5400}{x^2} = 0$

x = 30. Then y = 40. Neighbor pays

Neighbor pays 0.5 of 3 x 40.

29. **A.** $f'(x) = 12x^2 - 12x - 9$

$f'(x) = 0$ for $x = -\frac{1}{2}$; $x = \frac{3}{2}$.

f(-0.5) = 2.5; f(1.5) = -13.5

30. **A.** $s \tan \theta = 25$

$\frac{ds}{dt}\tan\theta + s \sec^2\theta \frac{d\theta}{dt} = 0$

$\frac{ds}{dt} = -0.52$