Interschool Test B
Test #843

1. Write your school's name (no abbreviations) at the bottom of each page of the answer sheet.

2. Scoring for this test is 1 point apiece unless otherwise noted.

3. This test should only be worked on by students (no sponsors, nor chaperones), and it should only be worked on by students who are attending the convention.

4. You may use calculators, the Internet, or other portable electronic devices on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.

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School Name: _____

Please write your answer in the appropriate blank.

1. (a) _____ (b) _____ (c) _____ (d) _____ (e) _____ (f) _____

   (g) _____ (h) _____ (i) _____ (j) _____

2. _____ 7. _____

3. _____ 8. _____

4. _____ 9. _____

5. _____ 10. _____

6. _____

Multiple Choice Section –Please write your answer (A,B,C,D,or E) in the corresponding box.

1.	6.	11.	16.
2.	7.	12.	17.
3.	8.	13.	18.
4.	9.	14.	19.
5.	10.	15.	20.

11. _____ 15. _____

12. _____ 16. _____

13. _____ 17. _____

14. _____ 18. _____

School Name: _____

19. _____

20. _____

21. _____

22. _____

23. _____

24. _____

25. _____

26. _____

27. _____

28. _____

29. _____

30. _____

31. _____

32. (a) _____

(b) _____

(c) _____

33. _____

34. _____

35. _____

36. _____

37. _____

38. _____

39. _____

40. _____

School Name: _____

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This is the Interschool Test, Part B. In all of the following problems, the answer we are looking for is the answer we are looking for. Answers are worth one point apiece unless otherwise noted. Good luck, and have fun!

1) Fill in the blanks with base 10 numbers so that the following sentence is true:

In this sentence, the digit 1 appears (a)_____ times, the digit 2 appears (b)_____ times, the digit 3 appears (c)_____ times, the digit 4 appears (d)_____ times, the digit 5 appears (e)_____ times, the digit 6 appears (f)_____ times, the digit 7 appears (g)_____ times, the digit 8 appears (h)_____ times, the digit 9 appears (i)_____ times, and the digit 0 appears (j)_____ times.

2) Assuming that letters used as digits are different from letters used as letters, it is possible to construct a sentence of the form shown in question 1 using any integer base $b \geq 2$ except for one. What is the number for which it is not possible to write such a sentence?

3) The following is two Sudoku puzzles in one. Each small square contains a two-digit number. The tens digits of all the numbers are laid out in a standard Sudoku layout, as are the units digits. In addition, each of the numbers 11-99 (excluding multiples of 10) appears exactly once in the grid below. What is the sum of the numbers in the four corners of the grid?

59			18			84		
	_2				94		43	
1_		74		63				98
_6	9_		_5	7_	_2	6_		_1
6_		_2	2_		5_	_9		1_
_4		1_	_6	3_	_3		9_	_2
93				15		31		6_
	85		91				_9	
		47			69			25

4) HR WKOVK WKVJAI OD SM QSIWA VZ DVIHROMN TSAAP VKHRVNVMSA ASHOM DGYSKD. AVMSKP YAK POP DVI DHYPOD VZ HROD FOMP VZ HROMN. SMLUSL, GYDHOVM ZVYK OD HR ZVAAVUOMN "HV ZVYK DONMOZOTSMH PONOHD, URSH OD HR MYIJK HRSH OD TVMDWOTYVYDAL IODDOMN ZKVI HROD WSKSNKSWR?"

5) What follows are sequences with different initial values. The rule that defines each sequence is the same. Each sequence terminates after the first time the number 4 appears (and it always appears eventually).

1, 3, 5, 4
2, 3, 5, 4
3, 5, 4
4
5, 4
6, 3, 5, 4
7, 5, 4
8, 5, 4
9, 4
10, 3, 5, 4
25, 10, 3, 5, 4
100, 10, 3, 5, 4

What is the smallest positive integer for which the first appearance of the number 4 in its sequence is the sixth element of the sequence?

6) One hundred students are dividing a large amount of candy among themselves. Student #1 takes 1% of the candy, then Student #2 takes 2% of what remains, then Student #3 takes 3% of what remains, and so on until Student #100 takes 100% of what's left at the end. Which student gets the largest amount of candy?

7) A thousand wires hang on a very high tower, so high that you cannot see which tip belongs with which bottom. This is something you are interested in knowing. You have a battery and a light bulb which will light up if two wires connect it to the battery with appropriate polarity (the battery and bulb each have two contact points, one each of + and -). Wires may be tied together to form longer wires and you can see the bulb light up even if you are on the opposite end of the tower. Since the tower is so high, you want to minimize the number of times you have to climb the staircase, regardless of how much you have to do while you are at the top or bottom. What is the minimum number of times you have to go from one end (either top or bottom) to the other?

8) What do the following numbers have in common: 78, 45, $33\frac{1}{3}$? (Your sponsor would probably figure this out right away.)

9) The distance between town A and town B is 1000 miles. There are 1000 apples in town A, and they need to be delivered to town B. The available truck can carry 1000 apples at most. The driver of the truck has an addiction to apples, and must eat one at the start of every mile he travels whenever there are apples in his truck. What is the largest number of apples the truck driver can deliver to town B?

10) An irrational decimal fraction is created by concatenating the positive integers:

0.12345678910111213141516171819202122...

If d_n represents the n^{th} digit of the fractional part of this number, find the value of

$$d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10,000} \times d_{100,000} \times d_{1,000,000}$$

Multiple Choice Section

The multiple choice section of this test is a test about the multiple choice section of this test. All of the questions on the multiple choice section of the test refer only to the multiple choice section of the test. The multiple choice section of this test is open book - that is, you are allowed to have a copy of the multiple choice section of this test when you take the multiple choice section of this test. Because the multiple choice section of this test is so easy (it's an open book multiple choice test!), questions in this section are only worth one half of one point each.

1. The first question whose answer is B is question
   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) 5

2. The only two consecutive questions with identical answers are questions
   (A) 6 and 7
   (B) 7 and 8
   (C) 8 and 9
   (D) 9 and 10
   (E) 10 and 11

3. The number of questions with the answer E is
   (A) 0
   (B) 1
   (C) 2
   (D) 3
   (E) 4

4. The number of questions with the answer A is
   (A) 4
   (B) 5
   (C) 6
   (D) 7
   (E) 8

5. The answer to this question is the same as the answer to question
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

6. The answer to question 17 is
(A) C
(B) D
(C) E
(D) none of the above
(E) all of the above

7. Alphabetically, the answer to this question and the answer to the following question are
(A) 4 apart
(B) 3 apart
(C) 2 apart
(D) 1 apart
(E) the same

8. The number of questions whose answers are vowels is
(A) 4
(B) 5
(C) 6
(D) 7
(E) 8

9. The next question with the same answer as this one is question
(A) 10
(B) 11
(C) 12
(D) 13
(E) 14

10. The answer to question 16 is
(A) D
(B) A
(C) E
(D) B
(E) C

11. The number of questions preceding this one with the answer B is
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

12. The number of questions whose answer is a consonant is
(A) an even number
(B) an odd number
(C) a perfect square
(D) a prime
(E) divisible by 5

13. The only odd-numbered problem with answer A is
(A) 9
(B) 11
(C) 13
(D) 15
(E) 17

14. The number of questions with answer D is
(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

15. The answer to question 12 is
(A) A
(B) B
(C) C
(D) D
(E) E

16. The answer to question 10 is
(A) D
(B) C
(C) B
(D) A
(E) E

17. The answer to question 6 is
(A) C
(B) D
(C) E
(D) none of the above
(E) all of the above

18. The number of questions with answer A equals the number of questions with answer
(A) B
(B) C
(C) D
(D) E
(E) none of the above

19. The answer to this question is:
(A) A
(B) B
(C) C
(D) D
(E) E

20. Which of the following was a great mathematician? (I told you this test was easy!)
(A) Newton
(B) Euler
(C) Euclid
(D) Gauss
(E) All of the above

This is the end of the multiple choice section of this test - we now return you to your regularly scheduled test!

11) A d-digit positive integer is called a *one-child number* if exactly one of its sub-strings is divisible by d.

For example, 5671 is a 4-digit one-child number. Among all its sub-strings 5, 6, 7, 1, 56, 67, 71, 567, 671, and 5671, only 56 is divisible by 4. Similarly, 104 is a 3-digit one-child number because only 0 is divisible by 3.

How many one-child numbers are there that are less than 10,000?

12) Albert and Bill are playing a game. They have a pile of n coins (where $n > 1$). On his first turn, Albert can take any whole number of coins as long as he doesn't take the whole pile. On each successive turn, the player may take up to twice the number of coins that his opponent took on immediately prior to that. The winner is the player who takes the last coin.

For example, if there are 5 coins in the pile, if Albert takes 2, 3, or 4 coins, Bill can take all the remaining coins. If Albert takes 1 coin, then Bill will take 1 coin, and then Albert can only take up to 2 coins, then Bill will take all the remaining coins. Thus, there is no winning strategy for Albert for the case where n=5.

Define M(n) as the maximum number of coins that Albert can take on the first turn of a winning strategy for the game with n coins. M(n) is 0 if there is no winning strategy for Albert. What is the value of M(10)+M(20)+M(30)+M(40)+M(50)+M(60)?

13) A unitary divisor d of a number n is a divisor of n that has the property gcd(d, n/d) = 1. The unitary divisors of 4! are 1, 3, 8, and 24.

What is the sum of the squares of the unitary divisors of 2013?

14) For an integer $n \geq 4$, we define the *lower prime square root* of n, lps(n), as the largest prime number less than or equal to $\sqrt{n}$, and the *upper prime square root* of n, ups(n), as the smallest prime number greater than or equal to $\sqrt{n}$. Let us call an integer $n \geq 4$ *semidivisible*, if exactly one of lps(n) and ups(n) divide n. What is the sum of all semidivisible numbers less than 100?

15) Find the unique positive base-10 integer whose square has the form 1A2B3C4D5E6F7G8H9I0, where A, B, C, D, E, F, G, H, and I each represent a single digit (not necessarily distinct).

16) A *prime-proof* number is a positive integer that cannot be made prime by changing no more than one of its digits. For instance, 75 is not prime-proof, because it can be made prime by changing its units digit to 9. What is the smallest prime-proof number?

17) The number, 1406357289, is a 0 to 9 *pandigital number* because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property.

Let d_1 be the 1st digit, d_2 be the 2nd digit, and so on. In this way, we note the following:

- $d_2d_3d_4$=406 is divisible by 2
- $d_3d_4d_5$=063 is divisible by 3
- $d_4d_5d_6$=635 is divisible by 5
- $d_5d_6d_7$=357 is divisible by 7
- $d_6d_7d_8$=572 is divisible by 11
- $d_7d_8d_9$=728 is divisible by 13
- $d_8d_9d_{10}$=289 is divisible by 17

Find the sum of all 0 to 9 pandigital numbers with this property.

18) What is the largest base-10 palindrome that is the product of two 3-digit numbers?

19) If all of the permutations of the digits of 1234567890 are written in numerical order, what is the number that appears in the one-millionth place on the list?

20) You are to write a positive integer as your answer for this question. Anything other than a positive integer will be treated exactly the same as if you had not answered this question. This question will be scored as follows: Let N be the second largest answer that any team gives to this question, and let your answer be n. If $n > N$, you will get 0 points for this question. If $n \leq N$, you will score $\frac{n}{N}$ points.

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The answers to this next section all begin with the letter O.

21) What is a word meaning "the day after tomorrow?"

22) What is the name of the symbol #?

23) This Shakespearian play is also the name of a board game.

24) This is the last name of the reporter who is friends of Michelangelo, Donatello, Raphael, and Leonardo.

25) This is the birth name of X-Men character Storm.

We are now back to having answers that do not necessarily begin with the letter O.

26) How many positive 15-digit integers in base 10 are there consisting only of the digits 1, 2, 3, 4, 5, (not all of these digits will necessarily get used) where adjacent digits always differ by exactly 1? For example, in the 4-digit case, 3234 is an integer that would work because $|3 - 2| = 1$, $|2 - 3| = 1$, and $|3 - 4| = 1$.

27) Find all positive integer palindromes in base 10 that does not have 1 as a digit, but where the product of the digits is 3 less than three times the sum of the digits.

28) Find the number of non-degenerate triangles which have sides of integer length and have a perimeter equal to 2013.

29) A right rectangular prism with dimensions of $289 \times 767 \times 920$ is made by gluing together 203929960 unit cubes of dimensions $1 \times 1 \times 1$. A space diagonal of this prism will pass through how many $1 \times 1 \times 1$ unit cubes?

30) The positive integers are arranged in a triangular array as shown below:

1	3	6	10	15	…
2	5	9	14	…	
4	8	13	…		
7	12	…			
11	…				
…					

The number 2013 is in the Rth row and Cth column. Find the value of C ÷ R.

31) If $\frac{x^2+y^2}{x^2-y^2} + \frac{x^2-y^2}{x^2+y^2} = 10$, then find the value of $\frac{x^8+y^8}{x^8-y^8} + \frac{x^8-y^8}{x^8+y^8}$.

32) Diophantine Equations are fun! Find a solution in integers to the following equations and express your answer as an ordered triple (x, y, z):

A. $x^3 + y^3 + z^3 = 28$
B. $x^3 + y^3 + z^3 = 29$
C. $x^3 + y^3 + z^3 = 30$

33) Calculus is fun, too! A sequence of continuous functions f_n is defined on the closed interval $x \in [0, 2013]$ by $f_1(x) = 2014 - x$, and for $n > 1$, $f_n(x) = 2\int_0^x \sqrt{f_{n-1}(t)}\, dt$.
Let $F(x) = \lim_{n\to\infty} f_n(x)$. Evaluate: $\int_0^{2013} F(x)\, dx$

34) Don't forget about systems of equations—they're fun, too! Find all ordered triples (x, y, z) of real numbers that satisfy the system:

$$x^{\frac{1}{3}} - y^{\frac{1}{3}} - z^{\frac{1}{3}} = 16$$

$$x^{\frac{1}{4}} - y^{\frac{1}{4}} - z^{\frac{1}{4}} = 8$$

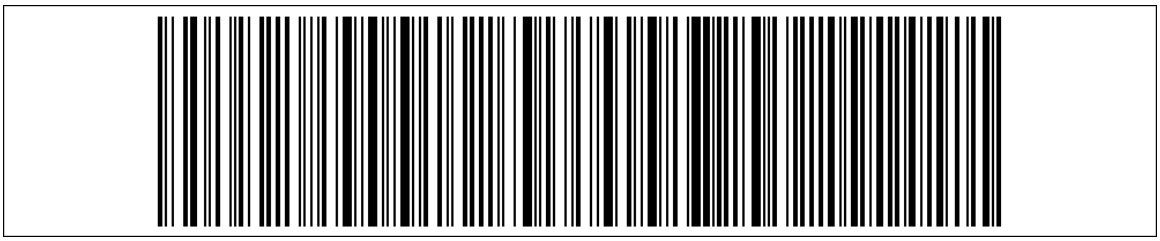
$$x^{\frac{1}{6}} - y^{\frac{1}{6}} - z^{\frac{1}{6}} = 4.$$

Express your answer as an ordered triple (x, y, z).

35) Find the largest divisor of vBkA0. (Wonder if it's larger than a googl…?)

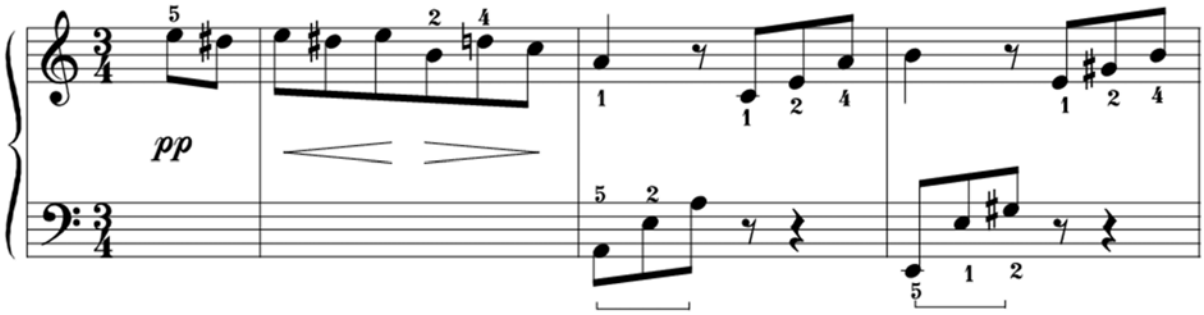
36) How many ways are there to tile a 15 × 4 rectangle with 2 × 1 dominoes such that the dominoes are not overlapping? For example, there is one way to tile a 2 × 1 or 1 × 2 rectangle with 2 × 1 dominoes. There are two ways to tile a 2 × 2 square, etc.

37) We've probably all seen this before…



38) Starting to see more of these, too…

39) Seems like a recent Interschool Test somewhere incorporated sheet music—quick, what is the name of the catchy piece whose sheet music excerpt is shown below?



40) Who is the mayor of San Diego during the administration of this Interschool Test B?