Limits and Derivatives – Calculus
Mu Alpha Theta National Convention 2003

* For all questions, answer E. "NOTA" means none of the above answers is correct. *

For questions 1 and 2, use the following table:

x	f(x)	$f'(x)$	g(x)	$g'(x)$
-2	1	-5	0	5
-1	-3	-3	4	3
1	-5	1	6	-1

1. Find $\frac{d}{dx}(f^3(x))$ when x = -2

A) –7 B) -24 C) –15 D) 1 E) NOTA

2. Find $\frac{d}{dx}\left(\frac{2g(x)}{x^2+1}\right)$ when x = -1

A) -7 B) 7 C) -1 D) 1 E) NOTA

3. Given that $f(x) = \sqrt{r^2 - x^2}$, evaluate the following limit: $\lim_{x \to \frac{r}{2}} [f(x)]$

A) $\sqrt{3} \cdot |r|$ B) $\frac{\sqrt{3} \cdot |r|}{2}$ C) 0 D) Limit does not exist E) NOTA

4. It costs Jared $1.60 to produce one CD of his band's music. After a good night of jammin' his band can sell their CD's for $4.75 each. What is the ratio of profit to CD's sold as they approach selling 25 discs?

A) $ 3.15 B) $ 4.69 C) $ 6.35 D) $ 78.75 E) NOTA

5. Evaluate the following limit: $\lim_{x \to 0^-} \sqrt{s \cdot x(5-x)}$ (Assume that s is a negative real number.)

A) 5s B) –5s C) 0 D) $\frac{s}{5}$ E) NOTA

6. The position (p) of a falling rock is a quadratic function of time (t) given by the equation:

$$p(t) = \frac{1}{2} g \cdot t^2 + v_0 \cdot t + p_0$$

Acceleration due to gravity	g
Initial velocity	v_0
Initial position	p_0

What is the rock's velocity as a function of time? (Hint: $v(t) = p'(t)$)

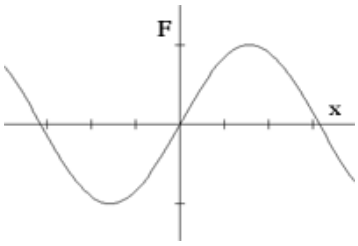
A) $v_0 \cdot t$

B) $\frac{1}{2} g \cdot t + v_0$

C) $g \cdot t + v_0$

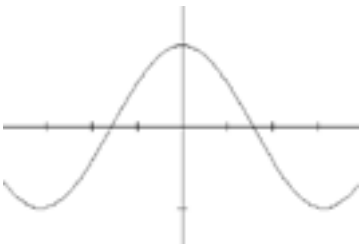
D) $g + v_0$

E) NOTA



7. The graph of a function F(x) is shown → →
What does the graph of the derivative of this function look like?

A)



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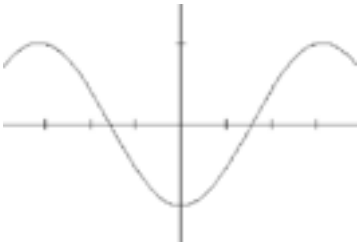

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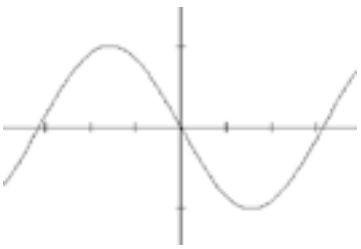
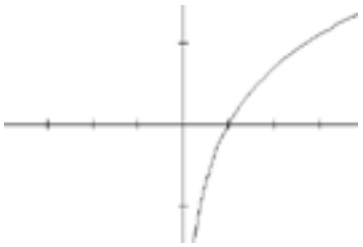


D)



E) NOTA

8. A function $\Phi(t)$ determines the velocity of a particle at any time *t*. The function is defined as:

$$\Phi(t) = \begin{cases} a \cdot t^4 & t \le 1 \\ b \cdot t^3 + 2 & t > 1 \end{cases}$$

What are the values of *a* and *b* that make this function differentiable at time *t* = 1?

A) The values do not exist B) a = 0, b = 0 C) a = 10, b = 7.5 D) a = -6, b = -8 E) NOTA

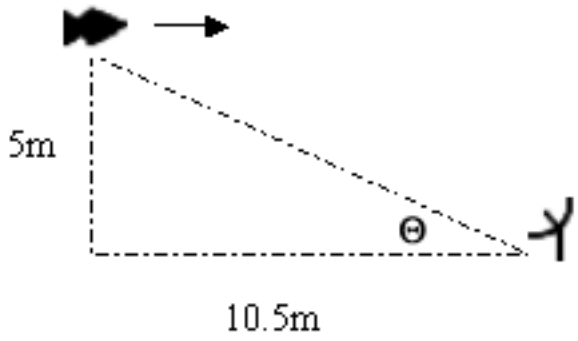
9. A microbiology scientist wants to use linear approximation to estimate the future amount of bacteria in a sample of pond water on day 3 of the experiment. She knows the following information:

Amount of Bacteria on day 1	1,270,000
Decay rate on day 1	26,000 per day
Average water temperature	25° C

What is the scientist's prediction (assuming she is correct)?

A) 1,218,000 B) 1,322,000 C) 1,244,000 D) 620,000 E) NOTA

10. A tomahawk cruise missile is flying at constant altitude of 5 miles over the desert. It is approaching a testing satellite that is 10.5 miles away on the ground. What is the instantaneous rate of change of the angle of elevation (Θ) to the nearest hundredth? (Note: the flight speed of the missile is 550 miles per hour.)



A) 42.70 radians/hour B) 20.33 radians/hour
C) 89.67 radians/hour D) Angle is constant
E) NOTA

11. The revenue function of selling x units is given by: $R(x) = 22.45x + .08x^2$. The cost function of producing x units is given by $C(x) = 15.33x + .12x^2$. What is the number of units that will maximize profit? (Answer to the nearest whole unit.)

A) 178 B) 473 C) 89 D) 59 E) NOTA

12. Find the slope of the tangent to the curve $x = t^3$ and $y = t^4$ when t = 3.

A) 3 B) $\frac{3}{4}$ C) $\frac{4}{3}$ D) 4 E) NOTA

13. Evaluate the following limit: $\lim_{n \to \infty} \left[n - \sqrt{n^2 - n} \right]$

A) $-\frac{1}{2}$ B) $\frac{1}{2}$ C) 0 D) $\frac{1}{\sqrt{2}}$ E) NOTA

14. An isosceles triangle is formed inside of a circle of radius *r*. One of the triangle's vertices is at the center of the circle and the other two are on the circle itself. What is the maximum area of such a triangle?

A) $\frac{r^2}{\sqrt{2}}$ B) $\frac{r^2}{2}$ C) $\frac{\sqrt{2}}{4} r^2$ D) $\frac{\pi \cdot r^2}{4}$ E) NOTA

15. A line determined by the equations x(t) = 3t + 5 and y(t) = t/8 is graphed in the Cartesian plane. What is the rate of change of the slope of this line?

A) $\frac{3}{8}$ B) $\frac{8}{3}$ C) 24 D) $\frac{1}{24}$ E) NOTA

16. Evaluate the following limit: $\lim_{x \to \infty} \left[(1 + x)^{\frac{1}{x+1}} \right]$

A) Limit does not exist B) 1 C) e D) 0 E) NOTA

17. $Y(x) = \int_0^x \left(\sin(t^2) \right) dt$. Which of the following statements are true?

A) Y is concave upward at x = π B) Y is maximized at x = 0 C) Y is increasing at x = -1

D) Y does not have a minimum value E) NOTA

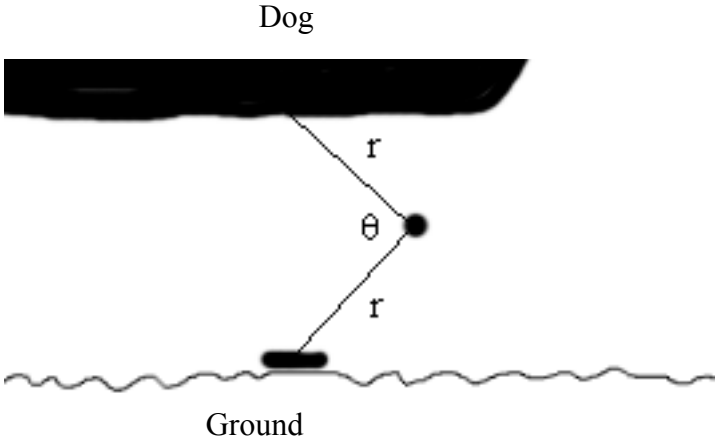
18. $y = ax + b\cos x$. What values of a guarantee the *nonexistence* of an inverse of y(x)?

A) (-b, b) B) [-b, b] C) $(-\infty,-b] \cup [b,\infty)$ D) (-1,1) E) NOTA

19. $Y(x) = F(G(x))$. Given that G is a parabolic function with vertex at (π, π) and F is differentiable everywhere. What is the slope of the line tangent to Y at $(\pi, Y(\pi))$?

A) Not enough information B) 0 C) 1 D) $-\pi$ E) NOTA

20. A robotic pet dog has mechanical legs that consist of 2 connected segments of equal length.



When the dog is in the "sitting down" process, the angle θ decreases at a rate of 0.5 radians per second. How quickly is the dog's body approaching the ground when θ = 150° and r = .45 meters? (Answer to nearest thousandth)

A) 0.067 meters/s B) 0.058 meters/s C) 0.075 meters/s D) 1.000 meters/s E) NOTA

21. Suppose that you have a calculator that can only perform addition, subtraction, multiplication, and division. In math class you need to calculate the cosine of 1.1 radians, so you decide to use a 4^{th} degree Maclaurin polynomial to approximate the end result. What is the accuracy of this approximation? (Round your answer to the nearest thousandth)

A) $0.454 < \cos(1.1) < 0.474$ B) $0.456 < \cos(1.1) < 0.469$ C) $0.443 < \cos(1.1) < 0.454$
D) $\cos(1.1) = 0.456$ E) NOTA

22. If f is continuous over [1, 3] and differentiable over (1, 3), suppose that $2 \le f'(x) \le 5$ for all x in [1, 3], and $f(1) = 4$. What is the maximum possible value that $f(3)$ can be?

A) 4 B) 5 C) 8 D) 14 E) NOTA

23. A roach crawls along the line tangent to the ellipse $\frac{x^2}{144} + \frac{y^2}{16} = 1$ at point $\left(6, \sqrt{12}\right)$. The roach crosses the y-axis at (0, a) and the x-axis at (b,0). What is the value of a – b ?

A) $10\sqrt{3}$ B) $\frac{15}{2}$ C) $\frac{15}{2} - 10\sqrt{3}$ D) $10\sqrt{3} - \frac{15}{2}$ E) NOTA

24. Find $\frac{d^2y}{dx^2}$ for the function $y = \frac{20x}{x^2+1} - \frac{1}{x}$.

A) $\frac{1}{x^2} - \frac{20\left(x^2-1\right)}{\left(x^2+1\right)^2}$ B) $\frac{x^2}{x^3-1} + \frac{2}{x^3}$ C) $\frac{x}{\sqrt{x^2+1}} - \frac{2}{x^3}$ D) $\frac{40x\left(x^2-3\right)}{\left(x^2+1\right)^3} - \frac{2}{x^3}$ E) NOTA

25. Which of the following is a possible solution to the differential equation $\frac{dy}{dx} - 4xy = 0$?

A) $y = e^{-4x} + 2$ B) $y = 2e^{-4x}$ C) $y = 2e^{2x^2}$ D) $y = e^{2x^2} + 2$ E) NOTA

26. A charged particle is initially at rest at the origin. When a magnetic field is initiated, the particle begins to spiral outward to infinity. The particle's net displacement from the origin is accelerating at a rate of 5 m/s^2. Also, the particle's angular velocity is 2 radians per second. What is the particle's acceleration in the x-axis direction after 6.5 seconds to the nearest tenth? (Note: the particle initially starts pointing in the positive y-axis direction. Also, it remains in the xy plane throughout the spiral.)

A) 56.3 m/s^2 B) 65.9 m/s^2 C) 161.0 m/s^2 D) 215.0 m/s^2 E) NOTA

27. A tank contains 500 liters of brine with 10 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. Approximately how much salt is in the tank after 10 minutes (to the nearest hundredth)?

A) 4.26 kg B) 5.33 kg C) 7.22 kg D) 8.19 kg E) NOTA

28. An equilateral triangle of side length s has one vertex on the y-axis and the other two on the x-axis. This triangle is rotated around the x-axis to form an object of volume V. If the side length is changing at a rate of $\frac{ds}{dt}$, at what rate is the volume changing?

A) $\frac{3\pi}{4}s^2 \cdot \frac{ds}{dt}$ B) $\frac{\pi}{4}s^3 \cdot \frac{ds}{dt}$ C) $\frac{s^2}{2} \cdot \frac{ds}{dt}$ D) $\left(\frac{ds}{dt}\right)^2$ E) NOTA

29. A particle's position in space is given by the equation $\vec{P}(t) = \left\langle \frac{\sin(2t)}{t}, e^{-t}, e^{t} \right\rangle$. Evaluate $\lim_{t \to 0^+} \vec{P}(t)$

A) $\langle 0,1,1 \rangle$ B) $\langle 2,1,1 \rangle$ C) $\langle 0,0,0 \rangle$ D) Limit does not exist E) NOTA

30. At what values of θ does the polar graph $r(\theta) = 2(1 - \cos(\theta))$ have a horizontal tangent? (For $0 < \theta < 2\pi$)

A) $\frac{\pi}{3}, \pi, \frac{4\pi}{3}$
B) $\frac{\pi}{3}, \frac{4\pi}{3}$
C) $\frac{2\pi}{3}, \frac{4\pi}{3}$
D) $0, \pi, 2\pi$
E) NOTA