
# Integration
# Mu, Round 2
# Test #412

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. Evaluate: $\int \frac{-x^2}{1+x^2}\,dx$

   (A) $x\ln(1+x^2) - \frac{x}{1+x^2} + C$ (B) $-x + \arctan x + C$

   (C) $-\ln|x| + \arctan x^2 + x + C$ (D) $-x^2 + x^2 \arctan x + C$ (E) NOTA

2. Given that $f(x) = \int_1^{\ln x} \frac{du}{u}$, evaluate $f'\left(\frac{1}{e}\right)$.

   (A) $e$ (B) $e^{-1}$ (C) $-e^{-1}$ (D) $-e$ (E) NOTA

3. Evaluate: $\lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{4n}{n^2+i^2}\right)$

   (A) $\frac{1}{\pi}$ (B) $\pi$ (C) $\pi^2$ (D) $\frac{\pi^3}{27}$ (E) NOTA

4. Evaluate: $\lim_{n\to\infty}\left(\int_0^n e^{-\frac{\sin^2 x}{4}} \cos x\,dx\right)$

   (A) $\frac{\sqrt{\pi}}{8}$ (B) $\frac{\sqrt{\pi}}{4}$ (C) $\frac{\sqrt{\pi}}{2}$ (D) $\sqrt{\pi}$ (E) NOTA

5. For $n > -1$, evaluate: $\int_1^e x^n \ln x\,dx$

   (A) $\frac{ne^{n+1}+1}{(n+1)^2}$ (B) $\frac{ne^{n+1}-1}{(n+1)^2}$ (C) $\frac{-ne^{n+1}+1}{(n+1)^2}$ (D) $\frac{ne^{n+1}+1}{(n-1)^2}$ (E) NOTA

6. Find the general solution $y(x)$ to the differential equation $\frac{d\left(\int_x^y dt\right)}{dy} = x$.

   (A) $y = \ln|1-x| + C$ (B) $y = -\ln|1-x| + C$

   (C) $y = -\ln|1+x| + C$ (D) $y = \ln|1+x| + C$ (E) NOTA

7. Evaluate: $\int \frac{1}{x\left(1+x^{\frac{1}{3}}\right)}\,dx$

(A) $\ln|x+1| - \ln\left|x^{\frac{2}{3}}+1\right| + C$ (B) $\ln|x-2| - 5\ln\left|x^{\frac{1}{4}}+1\right| + C$

(C) $\ln|x| - 2\ln\left|x^{\frac{1}{2}}+3\right| + C$ (D) $\ln|x| - 3\ln\left|x^{\frac{1}{3}}+1\right| + C$ (E) NOTA

8. Given that $a^2 > b^2$, evaluate: $2\int_{-a}^{a} \frac{\sqrt{(ab)^2-(xb)^2}}{a}\,dx$

(A) $\pi|ab|$ (B) $\frac{|ab|}{\pi}$ (C) $\frac{1}{\pi}\left|\frac{a}{b}\right|$ (D) $\frac{1}{\pi|ab|}$ (E) NOTA

9. Evaluate: $\int_0^{1.5\pi}\left(\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}\right)dx$

(A) 1 (B) $\frac{\sqrt{3}}{2}$ (C) 0.50 (D) 0 (E) NOTA

10. For this problem, a *pseudo light-year* is defined to equal 10 trillion kilometers. A particle is ejected downward from a spacecraft (consider this initial position zero) with an initial velocity of $-2000000\sqrt{10}$ kilometers per second, and a constant acceleration of 2 kilometers per second squared. How many seconds will it take for the particle to be a pseudo light-year away from its initial position?

(A) $1000000\sqrt{5}$ (B) $2000000\sqrt{5}$ (C) $1000000\sqrt{10}$ (D) $2000000\sqrt{10}$ (E) NOTA

11. For real number $u$, $-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}$ and $0 < \operatorname{arccot} u < \pi$. Evaluate:

$$\frac{\int_0^1 \operatorname{arccot}(1-x+x^2)\ dx}{\int_0^1 \arctan x\ dx}$$

(A) 3/2 (B) 1/2 (C) 2 (D) 1 (E) NOTA

12. Evaluate: $\int_1^{\sqrt{2}} x\arctan(x^2-1)\,dx$

(A) $\frac{\pi}{4} - \frac{1}{2}\ln 3$ (B) $\frac{\pi}{8} - \frac{1}{4}\ln 2$ (C) $\frac{\pi}{6} - \frac{1}{8}\ln 2$ (D) $\frac{\pi}{2} - \frac{1}{4}\ln 5$ (E) NOTA

13. Evaluate: $\int_0^{\pi} \sec^3 x \, dx$

(A) $\pi$ (B) $\pi^2$ (C) 1 (D) 0 (E) NOTA

14. Find the average value of $f(x) = \sin x$ on the interval $0 \leq x \leq 2\pi$.

(A) $-1$ (B) 0 (C) 1 (D) 2 (E) NOTA

15. Evaluate: $\int_0^{\pi/2} (1 + \tan x)^{-1} \, dx$

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\pi$ (E) NOTA

16. Find the volume of the solid whose base is the region with equation $4x^2 + 9y^2 \leq 36$, and cross sections to this solid perpendicular to the $x$-axis are in the shape of squares.

(A) 12 (B) 24 (C) 48 (D) 96 (E) NOTA

17. Evaluate: $\lim_{b \to 1^+} \int_1^b \frac{dx}{\sqrt{x(x-1)(b-x)}}$

(A) 0 (B) $2\sqrt{3}$ (C) $\pi$ (D) 5/2 (E) NOTA

18. Use Simpson's Rule with six equally-spaced partitions on the $x$-axis to approximate the area below the graph of $y = -x^2 + 8x - 12$ and above the $x$-axis.

(A) 25/3 (B) 28/3 (C) 29/3 (D) 32/3 (E) NOTA

19. How many of the following is/are true?

- All Riemann integrable functions are continuous.
- A continuous function is Riemann integrable.
- For a decreasing, concave-down function $f$ on the interval $x \in [a, b]$, and for integer $n \geq 2013$, $\sum_{i=0}^{n-1} f(a + id) \geq \sum_{i=1}^{n} f(a + id)$, where $d = (b - a)/n$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

20. Find the volume of the torus generated by revolving the circular region $(x - 5)^2 + y^2 \leq 1$ around the $y$-axis.

(A) $2\pi^2$ (B) $5\pi^2$ (C) $10\pi^2$ (D) $16\pi^2$ (E) NOTA

21. For $x \in \left[0, \frac{3}{4}\right]$, let $f(x) = \frac{x}{1+\frac{x}{1+\frac{x}{1+\ddots}}}$ and evaluate $\int_0^{3/4} f(x)\,dx$.

(A) $1/8$ (B) $5/24$ (C) $7/24$ (D) $9/24$ (E) NOTA

22. Evaluate: $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

(A) $\frac{\pi}{8}\ln 2$ (B) $\frac{5\sqrt{3}}{24}$ (C) $\frac{3}{4}$ (D) $\frac{\pi^2}{9}\ln 8$ (E) NOTA

23. If $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$, find the value of $\sin\sqrt{I}$.

(A) $1/2$ (B) $0$ (C) $\sqrt{2}/2$ (D) $1$ (E) NOTA

24. Find the area of one petal of the polar graph $r = 2\cos(6\theta)$.

(A) $\pi/8$ (B) $\pi/6$ (C) $\pi/4$ (D) $\pi/2$ (E) NOTA

25. A sequence $a_1, a_2, \ldots$ is explicitly defined as $a_n = \int_0^{\pi} \cot\left(\frac{x}{2}\right)\sin(nx)\,dx$. If $S = \sum_{n=1}^{2013} a_n$, find the value of $\cos S$.

(A) $1/2$ (B) $0$ (C) $1$ (D) $-1$ (E) NOTA

26. Let $f$ and $g$ be Riemann integrable even and odd functions, respectively. If $\int_0^a f(x)\,dx = 6$ and $\int_{-a}^0 g(x)\,dx = 8$, find the value of $\int_{-a}^a (2f(x) - 3g(x))\,dx$.

(A) $-12$ (B) $0$ (C) $12$ (D) $24$ (E) NOTA

27. If $I = \int_0^{\pi} \log_2(\sin x)\,dx$, find the value of $\cos I$.

(A) $1$ (B) $\sqrt{3}/2$ (C) $\sqrt{2}/2$ (D) $1/2$ (E) NOTA

28. Let $f$ be a nonnegative, continuous real-valued function on $x \in [0,1]$ such that $\int_0^1 \big(f(x)\big)^2\,dx = 1$. The maximum value of $\int_0^1 f(x)x^{2812}\,dx$ among all the possible values of $f$ is equal to $m/n$, where $m$ and $n$ are relatively prime positive integers. Find the value of $m+n$.

(A) 74 (B) 75 (C) 76 (D) 77 (E) NOTA

29. Evaluate: $\int x\sqrt{x+1}\,dx$

(A) $\frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{2}{5}(x+1)^{\frac{5}{2}} + C$ (B) $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

(C) $\frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$ (D) $\frac{1}{2}(x+1)^{2} - \frac{1}{4}(x+1)^{\frac{1}{4}} + C$ (E) NOTA

30. Evaluate: $\frac{d^2}{dx^2}\left(\int_1^{x^2} \ln(\sec t)\,dt\right)$

(A) $4x^2 \tan x^2 + 2\ln\sec x^2$ (B) $4x\tan x + 2\ln\sec x^2$

(C) $4x^2 \tan x^2 + 2\ln\sec x$ (D) $4x^3 \tan x^2 + 2\ln\sec x^2$ (E) NOTA