Limits & Derivatives
Mu, Round 1
Test #112

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. Let $f(x) = \frac{x^2-9}{x-3}$. What is $f(3)$?

   (A) 3 (B) 6 (C) 0 (D) Undefined (E) NOTA

2. Which of the following concepts is best associated with the slope of a function, the length of a curve, and the area of a region bounded by curves?

   (A) Derivatives (B) Limits (C) Integrals (D) Continuity (E) NOTA

3. Evaluate: $\lim_{x\to 2} \frac{\sin(x-2)}{2x-4}$

   (A) 0 (B) 0.50 (C) -1 (D) Undefined (E) NOTA

4. Find the derivative of $f(x) = \sin\left(\frac{1}{x^2+4}\right)$.

   (A) $\frac{2x\cos\left(\frac{1}{x^2+4}\right)}{(x^2+4)^2}$ (B) $-\frac{\cos\left(\frac{1}{x^2+4}\right)}{(x^2+4)^2}$ (C) $\cos\left(\frac{1}{x^2+4}\right)$ (D) $\frac{2x}{(x^2+4)^2}\sin\left(\frac{1}{x^2+4}\right)$ (E) NOTA

5. What is the value of $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)+1}$?

   (A) Undefined (B) -1 (C) 0 (D) 1 (E) NOTA

6. Let $f(x) = x^3 - 4x$. Which of the following is not a possible value of f'?

   (A) 1 (B) 0 (C) -4 (D) -6 (E) NOTA

7. Evaluate: $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

   (A) 1/3 (B) 3/8 (C) 1 (D) 1/2 (E) NOTA

8. Let $g(x) = \sqrt{4-x^2}$. For which of the following values of c does $\lim_{x\to c^-} g(x)$ exist but $\lim_{x\to c^+} g(x)$ not exist?

   (A) $c = 0$ (B) $c = 2$ (C) $c = -2$ (D) $c = 3$ (E) NOTA

9. If $h(x) = x^2\cos(x)$, what is the value of $h''\left(\frac{\pi}{2}\right)$?

(A) 2 (B) 2π (C) -2 (D) -2π (E) NOTA

10. Let f be a differentiable function where $f(1) = 2$, $f(2) = 5$, $f'(1) = 4$, and $f'(2) = 7$. Also, let $g(x) = f(x) + 5$. If $h(x) = g(f(x))$, find $h'(1)$.

(A) 4 (B) 7 (C) 16 (D) 28 (E) NOTA

11. Consider the function $f(x) = \frac{x^3+x^2-34x+56}{x^2+4x-12}$. Evaluate: $\lim_{h\to 0} f(2+h)$

(A) $-\frac{9}{4}$ (B) $-\frac{3}{2}$ (C) 0 (D) ∞ (E) NOTA

12. Find the limit as x approaches positive infinity of the function $c(x) = \frac{3x+40}{\sqrt{4x^2+1}-16}$.

(A) 0 (B) $\frac{3}{4}$ (C) $-\frac{5}{2}$ (D) ∞ (E) NOTA

13. For real number x, the Dirichlet Function is defined as $D(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$. D is an example of a function that is:

(A) Continuous and differentiable everywhere.
(B) Continuous everywhere.
(C) Continuous on part of its domain.
(D) Discontinuous everywhere.
(E) NOTA

14. The expression $\lim_{x\to 0} \frac{1-\cos(1-\cos x)}{x^4}$ is equal to m/n, where m and n are relatively prime positive integers. Find the sum of the digits of $m^2 + n^2$.

(A) 10 (B) 11 (C) 12 (D) 13 (E) NOTA

15. If $L = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$, find the sum of the digits of the greatest integer less than $1000L$.

(A) 20 (B) 18 (C) 10 (D) 9 (E) NOTA

16. Let g be a continuous and differentiable function defined by $g(x) = \sin(\pi x)$ if $x \le 2$ and $g(x) = ax^2 + b$ if x is any other real number. Find $a + b$.

(A) $\frac{\pi}{4}$ (B) $\frac{5}{4}\pi$ (C) $-\frac{3}{4}\pi$ (D) 0 (E) NOTA

17. Let $f(x) = \frac{1}{2}e^{2x-5}$. Find the maximum value of f''' on $0 \le x \le 3$.

(A) e (B) $2e$ (C) $3e$ (D) $4e$ (E) NOTA

18. Find the sum of all values of x for which the function $f(x) = \sqrt{|x^2 - 5x + 6|}$ is continuous, but not differentiable.

(A) -5 (B) 0 (C) 5 (D) 6 (E) NOTA

19. Evaluate: $\lim_{x\to 0} \frac{(1+\tan x)^{\frac{1}{x}} - e}{x}$

(A) -1 (B) Undefined (C) $-\frac{e}{2}$ (D) $-\frac{\pi}{3}\ln 2$ (E) NOTA

20. At what point does $y^2 + 2x^3 + 3xy = 0$ have a vertical tangent but no horizontal tangent?

(A) $\left(\frac{9}{8}, -\frac{27}{16}\right)$ (B) $(0,0)$ (C) $(1,-2)$ (D) $\left(\frac{3}{4}, -\frac{9+3\sqrt{3}}{8}\right)$ (E) NOTA

21. Find the derivative of $f(x) = \ln\left(\frac{x^4+1}{(\sin(4x)+2)^2}\right)$.

(A) $\frac{4x^3}{(\sin(4x)+2)^2} - \frac{8(x^4+1)\cos(4x)}{(\sin(4x)+2)^3}$ (B) $\ln(-\frac{x^3}{\cos(4x)})$

(C) $\frac{4x^3}{x^4+1} - \frac{8\cos(4x)}{\sin(4x)+2}$ (D) $\frac{4(x^4+1)\cos(4x)}{(\sin(4x)+2)^2} - \frac{4x^3}{\sin(4x)+2}$ (E) NOTA

22. Supposing that $y = 1 + x + x^2 + x^3 + \cdots$, with $|x| < 1$, which of the following expressions is equivalent to y'?

(A) y" (B) $y * y$ (C) y (D) $y!$ (E) NOTA

23. Evaluate: $\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx$

(A) e (B) $\frac{1}{4}$ (C) 1 (D) $\frac{3}{2}$ (E) NOTA

For problems 24 through 26, we examine the number e (~2.71828…). This number is defined by $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We will show that this limit converges using two sequences to help us out: $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = \left(1+\frac{1}{n}\right)^{n+1}$.

24. The first few steps in the proof are to show that a_n is an increasing sequence (that is, $a_{n+1} > a_n$) and that b_n is a decreasing sequence (that is, $b_{n+1} < b_n$). For both of these steps, which of the following inequalities will be the most helpful?

(A) AM-GM Inequality (B) Cauchy-Schwarz Inequality
(C) Triangle Inequality (D) Bernoulli's Inequality (E) NOTA

25. Next, if we can show that a_n and b_n converge to the same limit as n approaches positive infinity, we can define e to be that limit. Find a general formula for $b_n - a_n$.

(A) $\left(1+\frac{1}{n}\right)^n$ (B) $\left(1+\frac{1}{n}\right)^{n-1}$ (C) $\frac{a_n}{n!}$ (D) $\frac{1}{n}a_n$ (E) NOTA

26. It can be shown that $b_n > a_n$ for all positive integers n, and that $b_n - a_n < \frac{4}{n}$. Using this information, which of the following theorems can be best used to show that a_n and b_n converge to the same limit? In other words, that $\lim_{n\to\infty}(b_n - a_n) = 0$.

(A) Fermat's Last Theorem (B) Fundamental Theorem of Calculus
(C) Squeeze Theorem (D) Pythagorean Theorem (E) NOTA

27. Evaluate:

$$\lim_{s\to\infty}\frac{1}{s}\sum_{i=1}^{s}\sin\left(\frac{i\pi}{s}\right)$$

(A) $\frac{2}{\pi}$ (B) 2 (C) 2π (D) $\frac{\pi}{2}$ (E) NOTA

28. It can be shown, using the Delta-Epsilon definition of a limit, that $\lim_{x\to 4}\sqrt{x} = 2$. If $\varepsilon = 0.05$, which of the following values of δ would satisfy the definition?
1) $\delta = 0.2025$
2) $\delta = 0.1975$
3) $\delta = 0.05$

(A) 1, 2 and 3 (B) 2 and 3 only (C) 1 and 2 only (D) 3 only (E) NOTA

29. Let $h(x) = \tan(x^2)$. On the interval $0 \leq x \leq 2\pi$, how many points of discontinuity does h have?

(A) 12 (B) 13 (C) 14 (D) 15 (E) NOTA

30. Suppose f is a real-valued function defined on $|x| < 1$, where f is continuous at $x = 0$. Moreover, $f(0) = 1$ and if $|x| < 1$, then $(1 + x)f(x^2) = f(x)$. It can be shown that there is only one function f that satisfies the aforementioned properties. Find the sum of the digits of $f'\left(\frac{2012}{2013}\right)$.

(A) 27 (B) 24 (C) 21 (D) 18 (E) NOTA