Answer Key:

1. B
2. C
3. C
4. D
5. B

6. C
7. A
8. B
9. D
10. A

11. C
12. A
13. C
14. D
15. E

16. A
17. A
18. D
19. E
20. B

21. D
22. B
23. D
24. B
25. C

26. D
27. B
28. C
29. C
30. A

Solutions:

1. B: The 10th term is 9 positions after the first term. Therefore, since the first term is 23 and the terms increase by 4 each time, the 10th term is $23+9(4)=59$.

2. C: $\sum_{n=1}^{5}(9n-7)=(9\cdot1-7)+(9\cdot2-7)+(9\cdot3-7)+(9\cdot4-7)+(9\cdot5-7)=9(1+2+3+4+5)-5\cdot7=100$.

3. C: From the given information, the last term is $\frac{105-30}{5}=15$ positions after the first term. Therefore, there are 16 terms in the sequence.

4. D:

$$\sum_{n=0}^{5}2(n+3)^2=2(0+3)^2+2(1+3)^2+2(2+3)^2+2(3+3)^2+2(4+3)^2+2(5+3)^2$$

$$=2(9+16+25+36+49+64)=398.$$

5. B: In general, the sum of the first *n* positive integers is $\frac{n(n+1)}{2}$. Therefore, the sum of the first 800 positive integers is $\frac{800*801}{2}=400*801=320,400$.

6. C: In general, the sum of the first *n* positive integer perfect cubes is the square of the sum of the first *n* positive integers, the sum of the first *n* positive integers is $\frac{n(n+1)}{2}$, and the sum of the first *n* positive odd integers is n^2. Therefore, the sum of the first 100 positive integer perfect cubes, minus the sum of the first 5,050 positive odd integers, is $\left(\frac{100*101}{2}\right)^2-5,050^2=(50*101)^2-5,050^2=5,050^2-5,050^2=0$.

7. A: $\sum_{n=200}^{299}(2n+1)$ is the sum of the first 300 positive odd integers, minus the sum of the first 200 positive odd integers and therefore is 300^2-200^2.

$\sum_{n=100}^{199}(2n+1)$ is the sum of the first 300 positive odd integers, minus the sum of the first 200 positive odd integers and therefore is 200^2-100^2.

Therefore, $\frac{\sum_{n=200}^{299}(2n+1)}{\sum_{n=100}^{199}(2n+1)}=\frac{300^2-200^2}{200^2-100^2}=\frac{3^2-2^2}{2^2-1^2}=\frac{5}{3}$.

8. B: In general, the sum of the first *n* positive perfect squares is $\frac{n(n+1)(2n+1)}{6}$. Therefore, the sum of the first 60 positive perfect squares is $\frac{60*61*121}{6}=10*61*121=73,810$.

9. D: The x^5 term in the binomial expansion of $(x-1)^9$ is $\binom{9}{5}x^5(-1)^{9-5}=\frac{9*8*7*6*5}{1*2*3*4*5}x^5(-1)^4=9*2*7x^5=126x^5$, so the coefficient of this term is 126.

10. A: Since the sequence is *geometric* with first term 4 and second term 10, it has common *ratio* $\frac{10}{4}=\frac{5}{2}$; therefore, each term after the first term is $\frac{5}{2}$ times the previous term. Therefore, since the first term is 4, and the *n*th term is $n-1$ positions after the first term, it follows that the *n*th term is given by the algebraic expression $4\left(\frac{5}{2}\right)^{n-1}$.

11. C: Since the infinite geometric series has first and second terms 4 and $-\frac{1}{2}$, respectively, the common ratio is $\frac{-\frac{1}{2}}{4}=-\frac{1}{8}$. Since the sum of an infinite geometric series with a common ratio with absolute value less than one is equal to the first term, divided by the difference of 1 minus the common ratio, the sum of this infinite geometric series is $\frac{4}{1+\frac{1}{8}}=\frac{32}{9}$.

12. A: The 6th term of the binomial expansion of $(2x+3)^8$ is $\binom{8}{6-1}(2x)^{8-(6-1)}3^{6-1}=\binom{8}{5}(2x)^3 3^5=\frac{8*7*6*5*4}{1*2*3*4*5}*8x^3*243=8*7*8x^3*243=108{,}864x^3$, so the coefficient of this term is 108,864.

13. C: Let *r* represent the common ratio. Since the first two terms are each less than 32 and their sum is 32, they are both positive. So *r* is positive. Considering the terms of this infinite series in pairs, we see that 50 is also equal to the sum of another infinite geometric series, but with first term 32 and common ratio r^2. Therefore, $\frac{32}{1-r^2}=50$ which gives $1-r^2=\frac{16}{25}$ which gives $r^2=\frac{9}{25}$ which gives $r=\frac{3}{5}$ since *r* is positive.

14. D: Let *r* represent the common ratio. Since the first term is 1 and the sum of the infinite geometric series is $\frac{3}{2}$, we have $\frac{1}{1-r}=\frac{3}{2}$ which gives $r=\frac{1}{3}$. The cubes of these terms form another geometric series, but with first term 1 and common ratio $r^3=\left(\frac{1}{3}\right)^3=\frac{1}{27}$, so their sum is $\frac{1}{1-\frac{1}{27}}=\frac{27}{26}$.

15. E: In general, the sum of an infinite geometric series with a common ratio with absolute value less than one is equal to the first term, divided by the difference of 1 minus the common ratio. Therefore, from the given information, $54=\sum_{n=1}^{\infty}a\left(0.6\right)^{n}=\frac{0.6a}{1-0.6}$. Therefore,

$$\sum_{n=1}^{\infty}a\left(0.8\right)^{n}=\frac{0.8a}{1-0.8}=\frac{0.6a}{1-0.6}*\frac{1-0.6}{1-0.8}*\frac{0.8}{0.6}=54*2*\frac{4}{3}=144.$$

16. A: Since $\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}a^{2n+1}}{\left(2n+1\right)!}=0.1$, $0\le a\le\frac{\pi}{2}$, and in general $\sin x=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}x^{2n+1}}{\left(2n+1\right)!}$ for all real *x*, we have

$$\sum_{n=0}^{\infty}\frac{\left(-4\right)^{n}a^{2n+1}}{\left(2n+1\right)!}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}4^{n}a^{2n+1}}{\left(2n+1\right)!}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}\left(2a\right)^{2n+1}}{\left(2n+1\right)!}=\frac{1}{2}\sin\left(2a\right)=\sin a\cos a=0.1\sqrt{1-\left(0.1\right)^{2}}=\frac{3\sqrt{11}}{100}.$$

17. A: Let *D* represent the amount in dollars of each deposit. For $k=0,1,2,3,\ldots 20$, at time 21 years from now, the deposit *k* years from now will have earned annual compound interest of 3% for $21-k$ years.

In general, the sum of a finite geometric series is $\frac{a\left(1-r^{n}\right)}{1-r}=\frac{a\left(r^{n}-1\right)}{r-1}$, where *a* is the first term, *n* is the number of terms, and *r* is the common ratio.

Therefore, since an account balance of \$100,000 is needed 21 years from now, we have

$$100{,}000=D\sum_{k=0}^{20}\left(1.03\right)^{21-k}=D\sum_{m=1}^{21}\left(1.03\right)^{m}=\frac{1.03D\left[\left(1.03\right)^{21}-1\right]}{1.03-1}=\frac{D\left[\left(1.03\right)^{22}-1.03\right]}{0.03}.$$ Therefore,

$$D=\frac{3{,}000}{\left(1.03\right)^{22}-1.03}.$$

18. D: We have $4=E\left(X\right)=\sum_{n=0}^{\infty}nP\left(X=n\right)=\sum_{n=0}^{\infty}np^{n}\left(1-p\right)$.

For any nonnegative integer *k*, let $S_k=\sum_{n=0}^{k}np^{n}\left(1-p\right)$. Then $pS_k=\sum_{n=0}^{k}np^{n+1}\left(1-p\right)$.

Subtracting these two equations gives

$$\begin{aligned}
&\left(1-p\right)S_k\\
&=\sum_{n=0}^{k}np^{n}\left(1-p\right)-\sum_{n=0}^{k}np^{n+1}\left(1-p\right)\\
&=\sum_{n=0}^{k}np^{n}\left(1-p\right)-\sum_{m=1}^{k+1}\left(m-1\right)p^{m}\left(1-p\right)\\
&=\sum_{m=0}^{k}mp^{m}\left(1-p\right)-\sum_{m=1}^{k+1}\left(m-1\right)p^{m}\left(1-p\right)\\
&=0p^{m}\left(1-p\right)-kp^{k+1}\left(1-p\right)+\sum_{m=1}^{k}p^{m}\left(1-p\right)\\
&=-kp^{k+1}\left(1-p\right)+\frac{p^{1}\left(1-p\right)\left(1-p^{k}\right)}{1-p}\\
&=-kp^{k+1}\left(1-p\right)+p\left(1-p^{k}\right).
\end{aligned}$$

Therefore, $\sum_{n=0}^{k} np^n(1-p) = S_k = -kp^{k+1} + \frac{p(1-p^k)}{1-p}$. Therefore, since it is given that $0<p<1$,

$4 = \sum_{n=0}^{\infty} np^n(1-p) = \lim_{k\to\infty}\left[-kp^{k+1} + \frac{p(1-p^k)}{1-p}\right] = 0 + \frac{p(1-0)}{1-p} = \frac{p}{1-p}$. Solving for p gives $p = \frac{1}{\frac{1}{4}+1} = 0.8$.

19. E: The idea is to express the given series as a telescoping series, so that most of the terms cancel out. Observe that $\frac{0.5n+2}{0.5n+1} - \frac{0.5n+3}{0.5n+2} = \frac{n+4}{n+2} - \frac{n+6}{n+4}$.
Therefore, for every integer k greater than or equal to 1,
$\sum_{n=1}^{k}\left(\frac{0.5n+2}{0.5n+1} - \frac{0.5n+3}{0.5n+2}\right) = \sum_{n=1}^{k}\left(\frac{n+4}{n+2} - \frac{n+6}{n+4}\right) = \frac{5}{3} + \frac{6}{4} - \frac{k+5}{k+3} - \frac{k+6}{k+4}$.
Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{0.5n+2}{0.5n+1} - \frac{0.5n+3}{0.5n+2}\right) = \lim_{k\to\infty}\left(\frac{5}{3} + \frac{6}{4} - \frac{k+5}{k+3} - \frac{k+6}{k+4}\right) = \lim_{k\to\infty}\left(\frac{5}{3} + \frac{6}{4} - \frac{1+\frac{5}{k}}{1+\frac{3}{k}} - \frac{1+\frac{6}{k}}{1+\frac{4}{k}}\right) = \frac{5}{3} + \frac{6}{4} - \frac{1+0}{1+0} - \frac{1+0}{1+0} = \frac{7}{6}.$$

(Note: the answer is not just $\frac{5}{3} + \frac{6}{4} = \frac{19}{6}$, because the two surviving terms that depend on k do not converge to 0 as k goes to infinity.)

20. B: Integration is not required to solve this problem. In general, the sum of the first n positive integer perfect cubes is the square of the sum of the first n positive integers, and the sum of the first n positive integers is $\frac{n(n+1)}{2}$. Therefore, from using the properties of sums and limits, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left[3\left(\frac{k}{n}\right)^3 + \frac{k}{n}\right]$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3k^3}{n^4} + \frac{k}{n^2}\right)$$

$$= 3\lim_{n\to\infty}\frac{\sum_{k=1}^{n}k^3}{n^4} + \lim_{n\to\infty}\frac{\sum_{k=1}^{n}k}{n^2}$$

$$= 3\lim_{n\to\infty}\frac{\left[\frac{n(n+1)}{2}\right]^2}{n^4} + \lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}$$

$$= 3\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{4} + \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}$$

$$= \frac{3(1+0)^2}{4} + \frac{1+0}{2}$$

$$= \frac{5}{4}.$$

21. D: Since each stick has half the value of the stick just below it, the series we need to evaluate converges absolutely, so we can add the terms in any order we like. Also, in general, the sum of an infinite geometric series with a common ratio with absolute value less than one is equal to the first term, divided by the difference of 1 minus the common ratio. Therefore, the value of the stack 1 blue, 3 reds, 3 blues, 3 reds, 3 blues, 3 reds, 3 blues, … (from bottom to top) is $1-\sum_{n=0}^{\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\left(-\frac{1}{2^3}\right)^n=1-\frac{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}{1+\frac{1}{8}}=1-\frac{\frac{7}{8}}{\frac{9}{8}}=\frac{2}{9}$.

22. B: The idea is to express the given series as a telescoping series, so that most of the terms cancel out. Observe that $\frac{n}{n^4-2n^2+1}=\frac{n}{\left(n^2-1\right)^2}=\frac{1}{4}*\frac{4n}{(n-1)^2(n+1)^2}=\frac{1}{4}*\frac{(n+1)^2-(n-1)^2}{(n-1)^2(n+1)^2}=\frac{1}{4}\left[\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}\right]$.

Therefore, for every integer k greater than or equal to 3,

$$\sum_{n=3}^{k}\frac{n}{n^4-2n^2+1}=\frac{1}{4}\sum_{n=3}^{k}\left[\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}\right]=\frac{1}{4}\left[\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{k^2}-\frac{1}{(k+1)^2}\right].$$

Therefore, $\sum_{n=3}^{\infty}\frac{n}{n^4-2n^2+1}=\lim_{k\to\infty}\frac{1}{4}\left[\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{k^2}-\frac{1}{(k+1)^2}\right]=\frac{1}{4}\left(\frac{1}{2^2}+\frac{1}{3^2}-0-0\right)=\frac{13}{144}$.

23. D: Observe that for all non-negative integers n, $3\cdot 2^{n+1}+2=2\left(3\cdot 2^n+1\right)$. So in the partial product $\prod_{n=0}^{k}\frac{3\cdot 2^n+1}{3\cdot 2^n+2}$, k of the numerators are each paired with a (different) denominator twice that numerator, and the unpaired numerator and denominator are $3\cdot 2^k+1$ and $3\cdot 2^0+2=5$, respectively.

Therefore, for all non-negative integers k, $\prod_{n=0}^{k}\frac{3\cdot 2^n+1}{3\cdot 2^n+2}=\frac{3\cdot 2^k+1}{5}\left(\frac{1}{2}\right)^k=\frac{3+2^{-k}}{5}$.

Therefore, $\prod_{n=0}^{\infty}\frac{3\cdot 2^n+1}{3\cdot 2^n+2}=\lim_{k\to\infty}\frac{3+2^{-k}}{5}=\frac{3+0}{5}=\frac{3}{5}$.

24. B: For any positive integer n, the probability that Mike wins on his nth turn is the probability that Mike and Nathan each get tails on their first $n-1$ turns, and Mike gets heads on his nth turn. Therefore, the probability that Mike wins is $\sum_{n=1}^{\infty}(1-p)^{n-1}\left(\frac{1}{2}\right)^{n-1}p=\frac{p}{1-\frac{1}{2}(1-p)}=\frac{2p}{1+p}$. Since this winning probability is given as $\frac{4}{7}$, we have $\frac{2p}{1+p}=\frac{4}{7}$ which gives $14p=4+4p$ which gives $p=\frac{2}{5}$.

25. C: The probability of rolling a sum of 8 or more on a given roll of a pair of dice is $\frac{5+4+3+2+1}{6^2}=\frac{5}{12}$.
For any positive integer n, the probability that Joyce wins on her nth turn is the probability that Joyce and

Lynn each roll sums of less than 8 on their first $n-1$ turns, and Joyce rolls a sum of at least 8 on her *n*th turn. Therefore, the probability that Joyce wins is $\sum_{n=1}^{\infty}\left(1-\frac{5}{12}\right)^{2(n-1)}\left(\frac{5}{12}\right)=\frac{\frac{5}{12}}{1-\left(1-\frac{5}{12}\right)^2}=\frac{1}{1+\left(1-\frac{5}{12}\right)}=\frac{12}{19}$.

26. D: For any positive integer *n*, the probability that Beth wins on her *n*th turn is the probability that Alice rolls non-6's on her first *n* turns, Beth and Cherise each roll non-6's on their first $n-1$ turns, and Beth rolls a 6 on her *n*th turn. Therefore, the probability that Beth wins is

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{6}\right)^{3n-2}\left(\frac{1}{6}\right)=\frac{\frac{1}{6}\left(1-\frac{1}{6}\right)}{1-\left(1-\frac{1}{6}\right)^3}=\frac{\frac{5}{36}}{\frac{91}{216}}=\frac{30}{91}.$$

27. B: For any positive integer *n*, the probability that Alex wins on his *n*th turn is the probability that Alex, Boris, and Carlos each get tails on their first $n-1$ turns, and Alex gets heads on his *n*th turn. Therefore, since it is given that *p* is nonzero, the probability that Alex wins is

$\sum_{n=1}^{\infty}(1-p)^{3(n-1)}p=\frac{p}{1-(1-p)^3}=\frac{p}{p^3-3p^2+3p}=\frac{1}{p^2-3p+3}$. Since this winning probability is given as $\frac{1}{2}$, we have $\frac{1}{p^2-3p+3}=\frac{1}{2}$ which gives $p^2-3p+1=0$ which gives $p=\frac{3\pm\sqrt{9-4}}{2}=\frac{3\pm\sqrt{5}}{2}$. Since the probability *p* of showing heads must not exceed 1, $p=\frac{3-\sqrt{5}}{2}$.

28. C: Consider the sets in pairs. Note that a match ends only after an even number of sets, since the requirement for winning the match is to win two more sets than the other team. For any positive integer *n*, the probability that team A wins the match after *n* complete pairs of sets is the probability that team A wins exactly one set out of each of the first $n-1$ pairs, and wins both sets in the *n*th pair. So the probability that team A wins the match is $\sum_{n=1}^{\infty}\left[2p(1-p)\right]^{n-1}p^2=\frac{p^2}{1-2p(1-p)}=\frac{p^2}{1-2p+2p^2}=\frac{p^2}{p^2+(1-p)^2}$. Since this match winning probability is given as $\frac{25}{169}$, we have $\frac{p^2}{p^2+(1-p)^2}=\frac{25}{169}$ which gives $\frac{1}{1+\left(\frac{1}{p}-1\right)^2}=\frac{25}{169}$ which gives $\left(\frac{1}{p}-1\right)^2=\frac{144}{25}$ which gives $p=\frac{1}{1\pm\sqrt{\frac{144}{25}}}$. Since the probability *p* of winning any given set cannot be negative, $p=\frac{1}{1+\sqrt{\frac{144}{25}}}=\frac{1}{1+\frac{12}{5}}=\frac{5}{17}$.

29. C: From the cosine series, we have $0.2=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\cos x$ where x is in radians. Therefore,

$$\sum_{n=0}^{\infty}\frac{(-9)^n x^{2n}}{(2n)!}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n 9^n x^{2n}}{(2n)!}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n (3x)^{2n}}{(2n)!}$$
$$=\cos(3x)$$
$$=\cos(2x+x)$$
$$=\cos(2x)\cos x-\sin(2x)\sin x \quad .$$
$$=\left(2\cos^2 x-1\right)\cos x-2\sin^2 x\cos x$$
$$=\left(2\cos^2 x-1\right)\cos x-2\left(1-\cos^2 x\right)\cos x$$
$$=4\cos^3 x-3\cos x$$
$$=4(0.2)^3-3(0.2)$$
$$=-0.568.$$

30. A: In general, for any real number x,

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$
$$=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots$$
$$=\frac{1}{2}\left(\frac{2x}{1!}+\frac{2x^3}{3!}+\frac{2x^5}{5!}+\dots\right)$$
$$=\frac{1}{2}\left[\left(\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\dots\right)-\left(\frac{1}{0!}-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\dots\right)\right]$$
$$=\frac{1}{2}\left\{\left(\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\dots\right)-\left[\frac{1}{0!}+\frac{-x}{1!}+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\frac{(-x)^5}{5!}+\dots\right]\right\}$$
$$=\frac{1}{2}\left(e^x-e^{-x}\right).$$

Therefore, $\sum_{n=0}^{\infty}\frac{(\ln 2.5)^{2n+1}}{(2n+1)!}=\frac{1}{2}\left(e^{\ln 2.5}-e^{-\ln 2.5}\right)=\frac{1}{2}\left(e^{\ln 2.5}-\frac{1}{e^{\ln 2.5}}\right)=\frac{1}{2}\left(2.5-\frac{1}{2.5}\right)=1.05$.