Open Probability & Combinatorics 2013 MAΘ National Convention

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. Abigail, Brandon, George, Danielle, and Ernie are waiting in line at the movie theater. In how many different ways can they queue up if Danielle must be in front of Brandon?

   (A) 24 (B) 60 (C) 36 (D) 48 (E) NOTA

2. Link is being chased by a dodongo and only has one heart container left. In his bomb bag he has four bomb flowers each with a three second fuse, five bombs each with a five second fuse, and three bombchus each with a two second fuse. He reaches into his bag and grabs one item, blindly throwing it at his attacker. If the dodongo will reach and fatally wound Link in four seconds, what is the probability that Link will survive this ordeal? Assume the blast from any item is equally strong and will stop the dodongo immediately after the fuse burns out.

   (A) $\frac{7}{12}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{5}{12}$ (E) NOTA

3. Tom and Jerry are having a competition to see who can amass the largest amount of money in one day. Using a standard analog clock, Tom earns \$7.50 each time the hour hand and minute hand form a 180° angle, while Jerry earns \$4 each time the hour hand and minute hand form a right angle. What is the absolute value of the difference of money earned between Tom and Jerry?

   (A) \$10 (B) \$22 (C) \$5 (D) \$11 (E) NOTA

4. Joe, Josh, and Alan are sitting in a room watching TV and thinking about mathematics. They ponder the probability P of at least two of the members in the room having the same birthday. Furthering this thought, how many people would they need to invite into the room to be guaranteed a value of P that is 50% or greater?

   (A) 20 (B) 21 (C) 22 (D) 23 (E) NOTA

5. If V denotes the number of vertices of a graph and E denotes the number of edges of a graph, what is the value of the product VE for the complete bipartite graph $K_{4,7}$?

   (A) 28 (B) 162 (C) 39 (D) 308 (E) NOTA

6. Given an undirected graph G, which of the following sets of degrees of vertices of G cannot represent a semi-Eulerian graph?

I. (1, 2, 3, 2, 4) II. (2, 1, 3, 2) III. (2, 2, 4, 0) IV. (2, 2, 4) V. (1, 3, 1)

(A) III only (B) I and III only

(C) II and V only (D) III and V only (E) NOTA

7. Given an undirected graph G, which of the following sets of degrees of vertices of G cannot represent an Eulerian graph?

I. (2, 4, 2) II. (4, 0, 2) III. (1, 3, 2, 1) IV. (2, 4, 4, 2) V. (1, 3, 1)

(A) III and V (B) I and IV (C) V only (D) II and V (E) NOTA

8. Bob has three indistinguishable six-pack drink carriers and six various bottles of soda. In how many ways can Bob distribute his soda into the carriers such that no carrier is empty? (Note: Ignore the orientation of the bottles in the carrier).

(A) 25 (B) 15 (C) 31 (D) 90 (E) NOTA

9. Using the details in the note from the previous question (#8), Bob encounters an endless supply of indistinguishable six-pack drink carriers and another six various bottles of soda. In how many ways can Bob distribute his six bottles of soda into nonempty carriers?

(A) 203 (B) 52 (C) 91 (D) 140 (E) NOTA

10. Find the steady state vector for the Markov chain represented by the transition matrix

$$\begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

(A) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$ (B) $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ (C) $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ (D) $\begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ (E) NOTA

11. In a complete directed graph K_n, of order $n \geq 2$, how many Hamiltonian cycles exist?

(A) $n!$ (B) $(n\text{-}1)!$ (C) $\frac{(n-1)!}{2}$ (D) $\frac{n!}{2}$ (E) NOTA

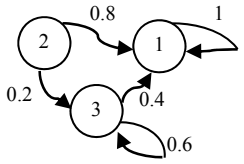
12. What is the sum of the seventh Catalan number and the chromatic number of a tree graph with n vertices, where $n > 1$?

(A) 427 (B) 429 (C) 431 (D) 433 (E) NOTA

13. The average number of aces for Novak Djokovic is 6 aces per match. What is the probability that Djokovic will hit exactly 8 aces in his next match? Assume a Poisson distribution ($P(k) = \frac{\lambda^k}{k!e^\lambda}$), use 2.7 as an approximation for e, and express your answer to the nearest tenth.

(A) 0.26 (B) 0.10 (C) 0.04 (D) 0.14 (E) NOTA

14. Given the following Markov chain, find its transition matrix.



(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0 & 0.2 \\ 0.4 & 0 & 0.6 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 0 & 1 \\ 0.2 & 0 & 0.8 \\ 0.6 & 0 & 0.4 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0.8 & 0.4 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.6 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0.8 & 0 & 0.2 \\ 0.6 & 0 & 0.4 \end{bmatrix}$ (E) NOTA

15. Using the transition matrix from Question #14, given an initial probability vector [0.2 0.2 0.6], what is the probability vector after 3 repetitions?

(A) [0.76 0 0.24] (B) [1 0 0]

(C) [0.8 0 0.2] (D) [0.856 0 0.144] (E) NOTA

16. Seven children at an elementary school begin to play a game. To start, each child removes his/her right shoe from his foot and places it into a garbage bag. One by one, the children blindly pick a shoe from the bag until no shoes remain unclaimed. What is the probability that no student receives his/her own shoe?

(A) $\frac{103}{280}$ (B) $\frac{53}{1008}$ (C) $\frac{1}{44}$ (D) $\frac{11}{160}$ (E) NOTA

17. Evaluate: $\binom{-7}{3} * \binom{3/2}{6}$

(A) $\frac{-143}{256}$ (B) $\frac{-147}{256}$ (C) $\frac{-145}{256}$ (D) $\frac{-149}{256}$ (E) NOTA

18. The famous Königsberg Bridges problem was first solved by whom?

(A) Hamilton (B) Horton (C) Pascal (D) Cayley (E) NOTA

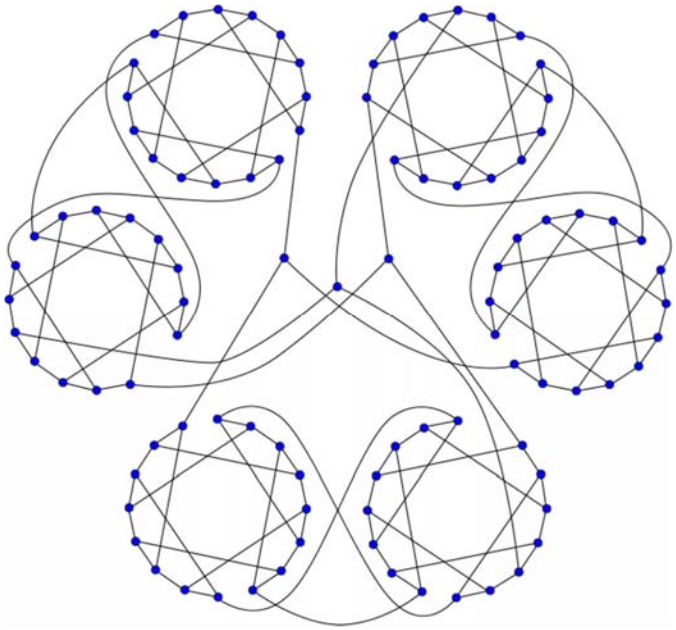
19. Two gunslingers wearing bandanas over their faces enter a bank at high noon and begin to rob the vault. Suddenly, cries erupt from the street that the sheriff is on his way. The robbers remove their bandanas and quickly blend in with the crowd. The sheriff decides to ask each of the patrons a series of questions and judge their reactions to determine their guilt. There are 30 people in the bank and the sheriff convicts a guilty person with probability 0.8 and an innocent person with a probability of 0.1. What is the probability that John Wayne is a bank robber given that the sheriff convicted him?

(A) $\frac{4}{11}$ (B) $\frac{11}{75}$ (C) $\frac{1}{15}$ (D) $\frac{4}{75}$ (E) NOTA

20. Kirby has a bag of coins, containing nine fair coins and a double-headed coin. If I randomly choose a coin from his bag and observe the coin flipping 5 heads in a row, what is the probability that the coin I chose is a double-headed coin?

(A) $\frac{1}{10}$ (B) $\frac{32}{41}$ (C) $\frac{41}{320}$ (D) $\frac{9}{320}$ (E) NOTA

21. The Horton graph is shown below.



Which of the following properties describe the Horton graph?

I. Cubic
II. Bipartite
III. Hamiltonian
IV. Eulerian
V. Planar
VI. Girth of 6

(A) I,III,VI only (B) I,II,V only (C) III,VI only (D) I,II,VI only (E) NOTA

22. Derrick, Steven, Luis, Brad, and three others attend a party. Upon entering, each person blindly draws a hat, either green or blue, and places it on his head. The seven partygoers must simultaneously guess the colors of their own hats or choose to pass. No communication regarding the color of one another's hat is allowed and each player can see all the hats except his own. The partygoers devise a group strategy that maximizes the probability P of at least one person guessing the color of his hat correctly and no one guessing incorrectly. What is the value of P?

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{7}{8}$ (E) NOTA

23. Given the following preference tables between men and women (1st being most preferable and 4th being the least preferable):

Men's Preference

	1st	2nd	3rd	4th
Henry	Angie	Darla	Billie	Cindy
Ian	Billie	Angie	Cindy	Darla
Joe	Angie	Cindy	Billie	Darla
Kyle	Angie	Cindy	Darla	Billie

Women's Preference

	1st	2nd	3rd	4th
Angie	Ian	Joe	Henry	Kyle
Billie	Joe	Kyle	Ian	Henry
Cindy	Ian	Kyle	Henry	Joe
Darla	Henry	Ian	Kyle	Joe

which of the following is a stable matching of 4 couples of men and women? Note: answers are expressed using the first initial of each of the individual's names, e.g. M⟶F (where M stands for a male's name and F for a female's name)

I. I⟶B, H ⟶A, J⟶C, K⟶D
II. I⟶B, J⟶A, H ⟶D, K⟶C
III. H ⟶D, J⟶B, K⟶C, I⟶A
IV. H ⟶A, I⟶C, K⟶D, J⟶B

(A) I only (B) III, IV only (C) II, IV only (D) II, III only (E) NOTA

24. John writes the 9 letters H, O, W, A, R, E, Y, O, U on index cards and places them into a hat. Mary blindly draws each of the 9 letters out of the hat, one at a time, and places them in a row. What is the probability that none of the words HOW, ARE, and YOU are spelled out when the 9 letters in a row are read from left to right consecutively?

(A) $\frac{4525}{4536}$ (B) $\frac{27779}{30240}$ (C) $\frac{463}{504}$ (D) $\frac{2381}{2592}$ (E) NOTA

25. Alex and Eb are both leaving for college at the end of summer and decide to try to meet up at the subway station. Feeling rushed and panicked, Eb tells Alex that she will arrive at the station no later than 8:30 A.M., no sooner than 7:58 A.M., and wait an undecided amount of time before boarding her subway. Alex mimics her arrival time and decides that he will only wait 11 minutes after he arrives at the station before boarding his subway. Given that the probability for each person showing up at the station is uniformly distributed, and given that the probability that the two will be waiting at the station at the same time is $\frac{65}{128}$, how many minutes does Eb wait for Alex before boarding her subway?

(A) 10 (B) 9 (C) 8 (D) 7 (E) NOTA

26. Tim has 7 various candy bars, including one Snickers bar, two Kit Kats, two Almond Joys, and two Twix bars. He decides to pass out all of the candy bars to his 4 kiddos, including Aaron, such that each person gets at least one candy bar. What is the probability that Aaron gets the Snickers bar?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$ (E) NOTA

27. Maxwell is walking on the Cartesian plane. He sets out from (-13, 7) and walks 17 units along a slope of $-\frac{8}{15}$ into quadrant IV. Nicholas tracks Maxwell's path on a grid drawn through all of the lattice points on the Cartesian plane. He counts the number of interiors of squares that Maxwell's path passes through and calls it *V*. What is the value of *V*?

(A) 23 (B) 22 (C) 21 (D) 20 (E) NOTA

28. Checkers leaves home for work, which is 8 units west and 9 blocks north of his home, and travels 17 units along a lattice grid. There is a fast food restaurant, which he loves, that is 4 units west and 3 units north of his home. If he passes by the restaurant, he can't resist the smell and always purchases a burger. What is the probability that Checkers buys a burger on his way to work?

(A) $\frac{252}{2431}$ (B) $\frac{735}{2431}$ (C) $\frac{420}{2431}$ (D) $\frac{882}{2431}$ (E) NOTA

29. Bob and Willis have an infinite supply of two types of coins; the first type is worth 11 cents and the second type is worth 17 cents. They write the positive integers from 1 to 160, inclusive, on a piece of paper and cut each integer out. These 160 identical cards are placed into a hat. If Bob and Willis blindly draw a card from the hat, what is the probability they can combine some of their coins to equal the amount written on the card?

(A) $\frac{1}{4}$ (B) $\frac{7}{16}$ (C) $\frac{15}{32}$ (D) $\frac{1}{2}$ (E) NOTA

30. James, Billy, Joel, Greg, Mike, Calvin, Chris, and Miguel are running the 100m hurdles. The track staff, arguably not very dependable, writes down the first initial of the runners' names in order to track the winners. When the presenter asks for the winners list, he realizes the staff's mistake. Not wanting to postpone the ceremony, the presenter cross-references the runners' initials with the winners list and reads out the names of the top 3 runners, guessing when needed. What is the probability that the presenter's results match with the exact results of the race?

(A) $\frac{3}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{2}{7}$ (E) NOTA