Theta Ciphering Questions and Solutions

P. For what values of K will the remainder of be the same when $x^3 – Kx^2 + 3x – 5$ is divided by $x - 2$ and $x + 3$.

Solution: By synthetic substitution, $8 – 4K + 6 – 5 = -27 – 9K – 9 – 5$; $k = -10$

1. Find x + y + z given that:

$$\log_2 (\log_3 (\log_4 x)) = 0$$
$$\log_3 (\log_2 (\log_4 y)) = 0$$
$$\log_4 (\log_3 (\log_2 z)) = 0$$

Solution: $\log_2 (\log_3 (\log_4 x)) = 0; \log_3 (\log_4 x) = 1; \log_4 x = 3; x = 64$
$\log_3 (\log_2 (\log_4 y)) = 0; \log_2 (\log_4 y) = 1; \log_4 y = 2; y = 16$
$\log_4 (\log_3 (\log_2 z)) = 0; \log_3 (\log_2 z) = 1; \log_2 z = 3; z = 8$ $\quad x + y + z = 88$

2. Find the smallest root of the equation $(x+2)(x+5)(x+9) – (x+2)(x+5)(2x+13) = 0$.

Solution: $(x+2)(x+5)(x+9) – (x+2)(x+5)(2x+13) = 0 \quad (x+2)(x+5)(x+9-2x-13)=0$
$(x+2)(x+5)(-x-4)=0 \quad x=-2, x=-5, x=-4$ The smallest is -5.

3. A hexagon is inscribed in a circle. Three of the sides are of length 3 and the other three sides are of length 4 and no two adjoining sides are the same length. What is the radius of the circle?

Solution: Extend the sides of the hexagon to form ΔABC which is equilateral.

$$\therefore\ 10 = \sqrt{y^2 + 25} \Rightarrow y^2 = 75 \quad r = \sqrt{4 + \frac{y^2}{9}} = \sqrt{4 + \frac{75}{9}} = \sqrt{\frac{111}{9}} = \frac{\sqrt{111}}{3}$$

4. Find $A^2 – B^2$ if $A = 2^{2003} + 2^{-2003}$ and $B = 2^{2003} – 2^{-2003}$.

Solution: $A^2 – B^2 = (A+B)(A-B) = (2^{2003} + 2^{-2003} + 2^{2003} – 2^{-2003})(2^{2003} + 2^{-2003} - 2^{2003} + 2^{-2003}) = (2(2^{2003}))(2(2^{-2003})) = 4$

5. How many linear arrangements (i.e. listings) of the six digits 1, 2, 3, 4, 5, 6, either start with a 2 or end with a 5 or both?

Solution: 2, __, __,__, __, __ or __, __, __, __, __, 5 or 2, __, __, __, __, 5

$5! + 5! – 4! = 120 + 120 – 24 = 216$

6. If the points (0, 0), (5, 3) and (8,0) are points on the circumference of a circle, determine the area of the circle.

Solution: Since (0, 0) and (8,0) are on the circumference then the point (4, y) is the center of the circle. The slope of the line between (5, 3) and (8, 0) is m = -1. The midpoint between (5, 3) and (8, 0) is (6.5, 1.5). The slope of the line between the midpoint and the center then is m = 1.

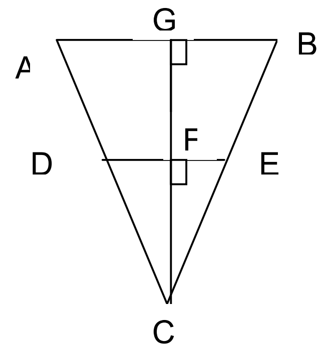
$$\frac{1.5 - y}{6.5 - 4} = 1 \quad 1.5 - y = 2.5 \quad y = -1$$

The radius of the circle is $\sqrt{4^2 + (-1)^2} = \sqrt{17}$ Area = 17π

7. Simplify by rationalizing the denominator: $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$

Solution: $\dfrac{1}{1+\sqrt{2}+\sqrt{3}} \cdot \dfrac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}} = \dfrac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2+\sqrt{2}-\sqrt{6}}{4}$

8. A water tank is in the shape of an inverted right circular cone. The radius of its base is 16 feet, and its height is 96 feet. What is the height, in feet, of the water in the tank if the amount of water is 25% of the tank's capacity?



Solution: Consider the cross-section of the tank passing through the vertex of the tank and the center of the base. In the diagram, G is the center of the base and F is the center of the water level. The volume of the tank is $\pi(96)\left(\dfrac{16^2}{3}\right)$. Therefore the volume of the water is $\pi\,(96)\left(\dfrac{256}{12}\right) = 2048\pi$. In the diagram, we see that $\triangle CFE \sim \triangle CGB$. If r is the radius of the top level of water and h be the height of the water, we get $\dfrac{r}{h} = \dfrac{16}{96} = \dfrac{1}{6}$, so $r = \dfrac{h}{6}$. Since the volume of the water is $2048\pi = \dfrac{\pi r^2 h}{3} = \dfrac{\pi h^3}{108}$ we have $h^3 = 108(2048)$ so h = $48\sqrt[3]{2}$

9. If 2 $\log_3$(x-2y) = $\log_3$ x + $\log_3$ y, find $\dfrac{x}{y}$.

Solution: 2 $\log_3$(x-2y) = $\log_3$ x + $\log_3$ y; $\log_3$ $(x-2y)^2$ = $\log_3$ (xy); $(x-2y)^2$ = xy;
$x^2 - 4xy + 4y^2 - xy = 0$; $x^2 - 5xy + 4y^2 = 0$; (x – 4y)(x – y) = 0; x = 4y or x = y; $\dfrac{x}{y}$=4 or $\dfrac{x}{y}$=1

If $\dfrac{x}{y}$=1, then x = y. Checking by substituting, 2$\log_3$(x - 2x) = $\log_3$ x + $\log_3$ x $\Rightarrow$
2 $\log_3$ (-x) = $\log_3$ x + $\log_3$ x would have to be positive for $\log_3$ x to exist, but then 2 $\log_3$ (-x) would not exist. So $\dfrac{x}{y} = 4$.

10. Two positive integers have a product of one billion and neither integer contains a zero. Find the smaller integer.

Solution: 1,000,000,000 = 10^9 = $(2\cdot 5)^9 = 2^9 \cdot 5^9 = (512)(1{,}953{,}125)$ Therefore, the smallest integer is 512.