Matrices & Determinants
Theta, Round 2
Test #433

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. Find the sum of a and b, given the following: $\begin{bmatrix} b & -a \\ -3 & 0 \end{bmatrix} + 2\begin{bmatrix} 2b & a+9 \\ a & -1 \end{bmatrix} = \begin{bmatrix} 25 & 16 \\ -7 & -2 \end{bmatrix}$

   (A) -18 (B) 3 (C) 5 (D) 8 (E) NOTA

2. Find the determinant of the following matrix: $\begin{bmatrix} 3 & 4 & 11 \\ 15 & -6 & -10 \\ -12 & 2 & 7 \end{bmatrix}$

   (A) -588 (B) -468 (C) 468 (D) 588 (E) NOTA

3. Let A = $\begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}$. Evaluate: $(A+B)^2 + 6B^{-1}$

   (A) $\begin{bmatrix} 87 & 77 \\ -2 & 51 \end{bmatrix}$ (B) $\begin{bmatrix} 82 & 159/2 \\ -1/3 & 148/3 \end{bmatrix}$
   (C) $\begin{bmatrix} 81 & 80 \\ 0 & 49 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -1/2 \\ -1/3 & 1/3 \end{bmatrix}$
   (E) NOTA

4. Let C = $\begin{bmatrix} 6 & 1 \\ 1 & 0 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 & -7 \\ 0 & -8 & 9 \end{bmatrix}$. After computing the product, what is the sum of the entries of C?

   (A) 29 (B) 49 (C) 0 (D) 154 (E) NOTA

5. Find the adjoint matrix of $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & -2 \\ 5 & 2 & -1 \end{bmatrix}$.

   (A) $\begin{bmatrix} 0 & 7 & -14 \\ -2 & -6 & 2 \\ -4 & -5 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -2 & -4 \\ 7 & -6 & -5 \\ -14 & 2 & 4 \end{bmatrix}$
   (C) $\begin{bmatrix} 0 & -7 & -14 \\ 2 & -6 & -2 \\ -4 & 5 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 2 & -4 \\ -7 & -6 & 5 \\ -14 & -2 & 4 \end{bmatrix}$

   (E) NOTA

6. What is the product of C_{32} and M_{14}, where C_{ij} denotes the *i*th row *j*th column cofactor and M_{ij} denotes the *i*th row *j*th column minor of matrix $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 4 & -1 & 0 \\ -1 & 2 & 1 & 5 \\ -1 & 4 & 6 & 1 \end{bmatrix}$?

(A) -2948 (B) -60 (C) 60 (D) 2948 (E) NOTA

7. Suppose that matrix A, shown below, has two eigenvalues of 1 and 10. What is the sum of the squares of the *other* eigenvalues of A?

$A = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}$

(A) 13 (B) 17 (C) 20 (D) 29 (E) NOTA

8. Which of the following is an eigenvector of $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$?

(A) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (B) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (C) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (D) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (E) NOTA

9. Let $A = \begin{bmatrix} 11 & 2 & 8 \\ 0 & 13 & 4 \\ 0 & 0 & 7 \end{bmatrix}$. What is the value of $\det(2A) - 7007\det(A^{-1})$?

(A) 999 (B) 1001 (C) 3993 (D) 8001 (E) NOTA

10. Given that the matrix $\begin{bmatrix} 5 & -2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & x \end{bmatrix}$ is singular, what is x?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

11. Which of the following matrices is an orthogonal matrix?

(A) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (E) NOTA

12. What is the rank of $\begin{bmatrix} 2 & 3 & -1 & 5 \\ 1 & 0 & 1 & 0 \\ 4 & 6 & 2 & 10 \end{bmatrix}$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

13. Suppose A is a 3 x 7 matrix. What is the sum of all possible values of the rank of A?

(A) 6 (B) 10 (C) 15 (D) 28 (E) NOTA

14. Find the area of a triangle whose vertices have coordinates (3, -8), (6, 8), and (-1, -6).

(A) 30 (B) 60 (C) 86 (D) 172 (E) NOTA

15. The first column of a 20 x 20 matrix contains the integer 9 in each of the rows. The second column is one-third of the first column. The third column is one-third of the second column, and so on. If this pattern continues, what is the trace of this matrix?

(A) $\frac{1}{2}(27 - 3^{-17})$ (B) 3^{-17} (C) $6(1 - 3^{-20})$ (D) $\frac{27}{2}$ (E) NOTA

16. What is A^{-1} if $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix}$?

(A) $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 2 & 0 \\ -3 & 3 & 1 \\ 6 & -7 & -2 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & -2 & 0 \\ -3 & 0 & 1 \\ 3 & -7 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & 1 \\ 6 & 7 & 2 \end{bmatrix}$

(E) NOTA

17. Evaluate: $\begin{vmatrix} -\ln e^4 & e^{\ln 3!} \\ e^{\ln 1} & \ln e^4 - \ln e^8 \end{vmatrix}$

(A) -22 (B) 10 (C) 13 (D) 16 (E) NOTA

18. Given the following system of equations:

$$\begin{cases} 371w + 157x + 261y - 385z = 157 \\ -100w + 380x - 467y + 75z = 380 \\ -54w + 383x + 72y - 440z = 383 \\ -34w + 262x + 161y + 272z = 262 \end{cases}$$

What is $(wy + xz)^5$?

(A) -1 (B) 0 (C) 1 (D) 32 (E) NOTA

19. Which of the following determinants, when set equal to 0, represents the equation of a circle passing through the points (1, -4), (3, -6), and (3, -2)?

(A) $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 17 & 1 & -4 & 1 \\ 45 & 3 & -6 & 1 \\ 13 & 3 & -2 & 1 \end{vmatrix}$ (B) $\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 1 & -4 & 17 \\ 1 & 3 & -6 & 45 \\ 1 & 3 & -2 & 13 \end{vmatrix}$

(C) $\begin{vmatrix} x+y & x & y & 1 \\ -3 & 1 & -4 & 1 \\ -3 & 3 & -6 & 1 \\ 1 & 3 & -2 & 1 \end{vmatrix}$ (D) $\begin{vmatrix} x^2+y^2 & x & y & 0 \\ 17 & 1 & -4 & 0 \\ 45 & 3 & -6 & 0 \\ 13 & 3 & -2 & 0 \end{vmatrix}$

(E) NOTA

20. Let $X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and A = $\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$. For $n \geq 0$, let $X_{n+1} = AX_n$. Which of the following is closest to the magnitude of the vector $X_{5,000,000,000,000}$?

(A) $\sqrt{2}$ (B) $\sqrt{5}$ (C) 5 (D) 3 (E) NOTA

21. Which of the following matrices is singular?

(A) $\begin{bmatrix} 6483 & 4444 & 3400 & 5220 \\ 8880 & 1523 & 6722 & 4218 \\ 8988 & 3528 & 5169 & 6120 \\ 9338 & 4334 & 1786 & 7449 \end{bmatrix}$ (B) $\begin{bmatrix} 5935 & 1868 & 3384 & 3902 \\ 6604 & 9901 & 6474 & 1486 \\ 9678 & 8790 & 7527 & 2552 \\ 8218 & 4484 & 3120 & 9693 \end{bmatrix}$

(C) $\begin{bmatrix} 4197 & 4086 & 9940 & 4482 \\ 2462 & 9059 & 4352 & 7770 \\ 1048 & 8648 & 5021 & 3574 \\ 4374 & 9152 & 3668 & 9481 \end{bmatrix}$ (D) $\begin{bmatrix} 5805 & 1310 & 2788 & 7252 \\ 3196 & 4449 & 2484 & 9178 \\ 6630 & 5148 & 9717 & 1810 \\ 9416 & 3496 & 2524 & 4733 \end{bmatrix}$

(E) NOTA

22. Find the sum of the elements of $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{2013}$.

(A) 8^{2013} (B) 8^{4026} (C) 2^{4025} (D) 2^{4027} (E) NOTA

23. Determine the number of 4×4 matrices, all of whose entries consist of a 0 or 1, that have an odd number of 1s in each row and each column.

(A) 64 (B) 128 (C) 512 (D) 32768 (E) NOTA

24. Let A be an $n \times n$ matrix where A^3 is equal to the $n \times n$ zero matrix. Let I equal the $n \times n$ identity matrix. How many of the following matrices are singular?

- A
- $I - A$
- $I + A$
- $I + A + \frac{1}{2}A^2$

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

25. What is the determinant of $A^2 + B^2$, if A and B are both $n \times n$ matrices such that $A \neq B$, $A^3 = B^3$, and $A^2B = B^2A$?

(A) -1 (B) 0 (C) 1 (D) 8 (E) NOTA

26. If A is a 50×50 matrix, B is a 20×50 matrix, and C is a 30×20 matrix, which of the following matrix products is well-defined?

(A) ABC (B) AB^TC (C) AB^TC^T (D) CA^2B (E) NOTA

27. A matrix B is called a *perfect square* if there is a matrix A such that $A^2 = B$. Which of the following matrices is a perfect square?

(A) $\begin{bmatrix} 3 & 16 \\ 14 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 16 \\ 16 & 4 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 64 \\ 16 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 11 & 7 \\ 14 & 18 \end{bmatrix}$ (E) NOTA

28. Let $n = det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $m = det\begin{pmatrix} 3c & 3d \\ 4a - c & 4b - d \end{pmatrix}$. Given that $n \neq 0$, if the value of $\left|\frac{m}{n}\right|$ is written as a common fraction in lowest terms, what is the sum of the numerator and denominator?

(A) 10 (B) 11 (C) 12 (D) 13 (E) NOTA

29. Suppose that $e^{\begin{vmatrix} 2 & 3 \\ -6 & -2 \end{vmatrix}} = \ln\left(\begin{vmatrix} e^e & 0 \\ \frac{1}{e} & a \end{vmatrix}\right)$. What is the value of $\frac{1}{\ln(a)}$?

(A) e^{-13} (B) e^{13} (C) e (D) 13 (E) NOTA

30. When the positive integers 1 to 9, inclusive, are used exactly once as elements of a 3×3 matrix, what is the largest possible value of the determinant of the matrix?

(A) 623 (B) 412 (C) 210 (D) 170 (E) NOTA