Theta Sequences and Series Solutions

1. $d = a_2 - a_1 = 4k + 1 - 2k = \boxed{2k+1}$.

2. $3 + 6 + ... + 60 = \frac{20}{2}(3+60) = \boxed{630}$.

3. $r = \frac{a_2}{a_1} = \frac{\left(\frac{2}{9}\right)}{\left(\frac{4}{3}\right)} = \boxed{\frac{1}{6}}$.

4. The first person shakes hands with the 31 others and leaves. The second person shakes hands with the remaining 30 and leaves, and so on. $31 + 30 + 29 + ... + 1 = \frac{31}{2}(1+31) = \boxed{496}$.

5. $a_n = a_1 + (n-1)d$; $a_1 = -10$, $d = -6 - (-10) = 4$, $a_{40} = -10 + 39 \cdot 4 = \boxed{146}$.

6. $S_n = \frac{a_1(r^n - 1)}{r-1}$; $a_1 = \frac{1}{3}$, $r = 2$, $S_{10} = \frac{1}{3}(2^{10} - 1) = \boxed{341}$.

7. $a_1 = 40$, $d = -5$, and $n = 1 + \frac{-100-40}{-5} = 29$. $S_{29} = \frac{29(40-100)}{2} = \boxed{-870}$.

8. The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \boxed{144}, \ldots$.

9. The number of rooms R on floor n is given by $R(n) = 5n - 2$.
   $\sum_{n=1}^{24}(5n-2) = 5 \cdot \frac{24(1+24)}{2} - 24 \cdot 2 = \boxed{1452}$.

10. The nth triangular number is the same as the sum of the first n positive integers. $T_{40} = \frac{40(1+40)}{2} = \boxed{820}$.

11. $a_6 = 6^2 - 2 \cdot 6 + 7 = 36 - 12 + 7 = \boxed{31}$.

12. The sum of the first n odd positive integers is n^2. $150^2 = \boxed{22500}$.

13. $a_1 = 27$ and $r = \frac{1}{3}$. $S = \frac{27}{1 - \frac{1}{3}} = \frac{81}{2} = \boxed{40.5}$.

14. On the nth bounce, the ball will rise $25(\frac{2}{5})^n$ meters. $25 \cdot \frac{2^6}{5^6} = \frac{2^6}{5^4} = \frac{2^{10}}{2^4 \cdot 5^4} = \frac{2^{10}}{10^4} = \frac{1024}{10000} = .1024 \text{ m} = \boxed{10.24 \text{ cm}}$.

15. The 32nd perfect cube is $32^3 = (2^5)^3 = 2^{15} = \boxed{32768}$.

16. Each successive square is one half the size of the previous square (in terms of perimeter), so each successive diagonal is half the size of the previous. The first diagonal is $2\sqrt{2}$, so the next is $\sqrt{2}$, the next is $\frac{\sqrt{2}}{2}$ and so on. The lengths of the diagonals form a geometric sequence with $r = \frac{1}{2}$ and $a_1 = 2\sqrt{2}$, so $S = \frac{2\sqrt{2}}{1 - \frac{1}{2}} = \boxed{4\sqrt{2}}$.

17. Thom will win the bet if Thom wins the game. Thom can win the game if triples are rolled on the 2nd, 4th, 6th, 8th, etc. trial (since Tom goes first). The probability of rolling triples on 3 six-sided dice is $P(T) = \frac{6}{6^3} = \frac{1}{36}$. Therefore, the probability that Thom will win is the infinite sum:
    $P(T)^C \cdot P(T) + (P(T)^C)^3 \cdot P(T) + (P(T)^C)^5 \cdot P(T) + \ldots = P(T) \cdot \frac{P(T)^C}{1 - (P(T)^C)^2}$
    $P(T)^C = 1 - P(T) = \frac{35}{36}$, so we have $\frac{1}{36} \cdot \frac{\frac{35}{36}}{1 - \frac{35^2}{36^2}} = \frac{35}{36^2(1 - \frac{35^2}{36^2})} = \frac{35}{36^2 - 35^2} = \frac{35}{(36-35)(36+35)} = \boxed{\frac{35}{71}}$.

18. The standard infinite geometric series formula works whenever $|r| < 1$, even if r is complex. $a_1 = 1$ and $r = (2+3i)^{-1}$, so we have $S = \frac{1}{1-(2+3i)^{-1}} = \frac{2+3i}{2+3i-1} = \frac{2+3i}{1+3i} = \frac{(2+3i)(1-3i)}{10} = \boxed{\frac{11-3i}{10}}$.

19. We are seeking the value of the product $2 \cdot 4 \cdot 6 \cdots 40 = 2^{20} \cdot (1 \cdot 2 \cdot 3 \cdots 20) = \boxed{2^{20} \cdot 20!}$.

20. $F_0 = 0, F_{-1} = 1, F_{-2} = -1, F_{-3} = 2, F_{-4} = -3, F_{-5} = 5, F_{-6} = -8, \ldots$. We can see that $F_{-n} = (-1)^{n+1} \cdot F_n$. $F_{15} = 610$ so $F_{-15} = \boxed{610}$.

21. $\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$.
$\sum_{n=1}^{35} n^2 = \frac{35(36)(71)}{6} = \boxed{14910}$.

22. The first 3 terms for the sequence $(2n-1)(n^2-n+1)$ are $(2-1)(1^2-1+1) = 1$, $(4-1)(2^2-2+1) = 9$, and $(6-1)(3^2-3+1) = 35$. $1 + 9 + 35 = \boxed{45}$.

23. On day n, he spends $10 \cdot (1.1)^{(n-1) \mod 9}$ dollars. June 6 is day 1, so June 27 is day 22. $(22-1) \mod 9 \equiv 3$ so he spends $\$10 \cdot 1.1^3 = \$10 \cdot 1.331 = \boxed{\$13.31}$.

24.

$$\begin{aligned}
3 + \frac{2}{3 - \frac{2}{x}} &= x \\
\frac{2x}{3x-2} &= x - 3 \\
(3x-2)(x-3) &= 2x \\
3x^2 - 13x + 6 &= 0 \\
x &= \frac{13 \pm \sqrt{169-72}}{6} \\
&= \frac{13 \pm \sqrt{97}}{6}
\end{aligned}$$

We can discard the negative solution since this fraction must be greater than 0, therefore we have $\boxed{\dfrac{13+\sqrt{97}}{6}}$.

25. We have a 5 term geometric sequence with $a_1 = 16$ and $a_5 = a_1 \cdot r^4 = 36$. $r^4 = \frac{36}{16} = \frac{9}{4}$, so $r = \left(\frac{9}{4}\right)^{\frac{1}{4}} = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$. The numbers are ordered from least to greatest with 16 being the least, so r must be positive. The sum $S_5 = \frac{a_1(r^4-1)}{r-1} + 36 = \frac{16(\frac{9}{4}-1)}{\frac{\sqrt{6}}{2}-1} + 36 = \frac{20}{\frac{\sqrt{6}}{2}-1} + 36 = \frac{20(\frac{\sqrt{6}}{2}+1)}{\frac{1}{2}} + 36 = \boxed{76 + 20\sqrt{6}}$.

26. Let $Q_n = An^2 + Bn + C$. Then $Q_1 = A + B + C = 10$, $Q_3 = 9A + 3B + C = 18$, and $Q_5 = 25A + 5B + C = 27$. $9Q_1 - Q_3 = 6B + 8C = 72$. $25Q_1 - Q_5 = 20B + 24C = 223$.

$$\begin{aligned}
6B + 8C &= 72 \\
20B + 24C &= 223
\end{aligned}$$

Multiply the top equation by 10 and the bottom equation by -3:

$$\begin{aligned}
60B + 80C &= 720 \\
-60B - 72C &= -669
\end{aligned}$$

$$\begin{aligned} 8C &= 51 \\ C &= \frac{51}{8} \end{aligned}$$

It follows that $B = \frac{1}{6} \cdot (72 - 51) = \frac{7}{2}$. $Q_3 - Q_1 = 8A + 2B = 8$, so $A = \frac{1}{8} \cdot (8 - 7) = \frac{1}{8}$.

Therefore, $Q_6 = 36 \cdot \frac{1}{8} + 6 \cdot \frac{7}{2} + \frac{51}{8} = \frac{36(1)+6(28)+51}{8} = \boxed{\frac{255}{8}}$.

27. $a_1 = a$, $a_2 = b = a \cdot r$, and $a_3 = 6 = a \cdot r^2$. $a + b = 1$ so $a + a \cdot r = 1$ and thus $r = \frac{1}{a} - 1$. $a \cdot r^2 = 6$ so $r^2 = \frac{6}{a}$.

$$\begin{aligned} r &= \frac{1}{a} - 1 \\ 6r &= \frac{6}{a} - 6 \\ 6r &= r^2 - 6 \\ r^2 - 6r - 6 &= 0 \\ r &= \frac{6 \pm \sqrt{36 + 24}}{2} \\ &= \boxed{3 \pm \sqrt{15}} \end{aligned}$$

28. If $f(x) = f(x-1) + 2$ then $f(x-1) = f(x) - 2$. Thus, $f(-2) = 5 - 2 = 3$, $f(-3) = 3 - 2 = 1$, $f(-4) = 1 - 2 = -1$ and $f(-5) = -1 - 2 = \boxed{-3}$.

29. $\sum_{n=0}^{\infty} \frac{n}{5^n} =$

$$\begin{array}{llll} \frac{1}{5} & +\frac{1}{5^2} & +\frac{1}{5^3} & +\ldots = \frac{\frac{1}{5}}{1-\frac{1}{5}} \\ & +\frac{1}{5^2} & +\frac{1}{5^3} & +\ldots = \frac{\frac{1}{5^2}}{1-\frac{1}{5}} \\ & & +\frac{1}{5^3} & +\ldots = \frac{\frac{1}{5^3}}{1-\frac{1}{5}} \\ & & & \ldots \end{array} = \frac{\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\left(1-\frac{1}{5}\right)^2} = \frac{5}{16}.$$

$\frac{5}{16} + \left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots\right) = \frac{5}{16} + \frac{1}{4} = \boxed{\frac{25}{16}}$.

30. $\sum_{n=1}^{2005} (n \cdot 2^n + 1) = 2005 + \sum_{n=1}^{2005} n \cdot 2^n$.

$\sum_{n=1}^{2005} n \cdot 2^n =$

$$\begin{array}{lllllll} 2^1 & +2^2 & +2^3+\ldots & +2^{2005} & = 2(2^{2005}-1) & = 2^{2006}-2 \\ & +2^2 & +2^3+\ldots & +2^{2005} & = 2^2(2^{2004}-1) & = 2^{2006}-2^2 \\ & & +2^3+\ldots & +2^{2005} & = 2^3(2^{2003}-1) & = 2^{2006}-2^3 \\ & & & & \ldots & \\ & & & +2^{2005} & = 2^{2005}(2^1-1) & = 2^{2006}-2^{2005} \end{array}$$

$= 2005 \cdot 2^{2006} - (2 + 2^2 + 2^3 + \ldots + 2^{2005}) = 2005 \cdot 2^{2006} - 2(2^{2005} - 1) = 2005 \cdot 2^{2006} - 2^{2006} + 2 = 2004 \cdot 2^{2006} + 2$.

The entire sum is equal to $2004 \cdot 2^{2006} + 2007$. To find the number of digits in the base 10 representation of this number, we can use the formula $\lfloor \log_{10}(N) \rfloor + 1$.

$\log_{10}(2004 \cdot 2^{2006} + 2007) \approx \log_{10}(2004 \cdot 2^{2006} + 2004) \approx \log_{10}\left(2004(2^{2006} + 1)\right) \approx \log_{10}\left(2004 \cdot 2^{2006}\right) \approx \log_{10}(2004) + 2006 \log_{10}(2) \approx \log_{10}(2000) + 2006 \cdot .301 \approx \log_{10}(2) + \log_{10}(1000) + 603.806 \approx .301 + 3 + 603.806 \approx 607.1$.

$1 + \lfloor 607.1 \rfloor = \boxed{608}$ digits.