2009 Alpha Individual Test Solutions

SOLUTIONS:

1. $m = \frac{5-3}{-3-2} = \frac{3-y}{2-4}$; $-15 + 5y = -4 \rightarrow y = \frac{11}{5}$ ***A***

2. $y = 2x^2 - 5x + 1 \rightarrow x = 2y^2 - 5y + 1$, $x - 1 + \frac{25}{8} = 2\left(y^2 - \frac{5}{2}y + \frac{25}{16}\right)$, $\pm\sqrt{\frac{8x+17}{16}} = y - \frac{5}{4}$; $y = \frac{5+\sqrt{8x+17}}{4}$, $x \geq -\frac{17}{8}$. ***D***

3. $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{49}{25} \rightarrow \sin 2x = \frac{24}{25} \rightarrow \cos^2 2x = 1 - \frac{576}{625}$; $\cos^2 = \pm\frac{7}{25}$, ***E***

4. $\log_{\sqrt[4]{3}} \log_{\sqrt[3]{10}} \log_2 1024 = \log_{\sqrt[4]{3}} \log_{\sqrt[3]{10}} 10 = \log_{\sqrt[4]{3}} 3 = 4$. ***B***

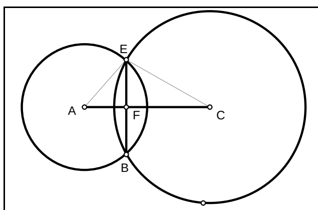
5. $\frac{1}{14}\begin{bmatrix} 5 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} X = \frac{1}{14}\begin{bmatrix} 5 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -5 & -2 \\ 39 & 52 \end{bmatrix}$; $X = \begin{bmatrix} 1 & 3 \\ 7 & 8 \end{bmatrix}$. ***C***

6. Looking at a few cases: 3,4,5;5,12,13;7,24,25, indicates that 3, 4, and 5 divides *P*. It can be proven using a case of even and odd values that show this to be true. **60** ***D***

7. Smithson: J + 8 or 1, 9, 17,... Jackson: K + 5 + 1 or 2,7,12,.. → 17 − 1 = 16. ***A***

S	1	9	<u>**17**</u>	25
J	2	7	12	***17***

8.



$EF^2 = EC^2 - x$ and $EF^2 = EA^2 - (14 - x)^2$;
$225 - x^2 = 169 - 196 + 28x - x^2$;
$28x = 252 \rightarrow x = 9$ and $225 - 81 = EF^2$ and $EF = 12$.

EB = 24 ***B***

9. $\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} \cdot \frac{\sqrt[3]{1}-\sqrt[3]{2}}{\sqrt[3]{1}-\sqrt[3]{2}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} \cdot \frac{\sqrt[3]{2}-\sqrt[3]{3}}{\sqrt[3]{2}-\sqrt[3]{3}} + \frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}} \cdot \frac{\sqrt[3]{3}-\sqrt[3]{4}}{\sqrt[3]{3}-\sqrt[3]{4}} = \frac{\sqrt[3]{1}-\sqrt[3]{2}}{1-2} + \frac{\sqrt[3]{2}-\sqrt[3]{3}}{2-3} + \frac{\sqrt[3]{3}-\sqrt[3]{4}}{3-4} =$

$-\sqrt[3]{1} + \sqrt[3]{2} - \sqrt[3]{2} + \sqrt[3]{3} - \sqrt[3]{3} + \sqrt[3]{4} = \sqrt[3]{4} - 1$. ***C***

10. $f(n) = 2,5,10,17 \rightarrow n^2 + 1$ for $n = 1, 2, 3,\ldots$; $f(100) = 10001$ ***D***

11. $3^{x^2-2xy} = 1$, $x^2 - 2xy = 0$; Solving for $x \rightarrow x(x - 2y) = 0$ and $x = 2y$ $(x \neq 0)$;
$2\log_3 x = \log_3(y + 3)$, $x^2 = y + 3$; Solving for $y \rightarrow y = x^2 - 3$. Subst. for $x \rightarrow y = 4y^2 - 3$ and $(4y + 3)(y - 1) = 0$.
$y = 1, -\frac{3}{4}$, $x = 2, 0, -3/2$. Only $y = 1$ is a solution. ***E***

12. $2x^3 - 7x^2 + kx - 2 = 0$. Let x_1, x_2, and x_3 be the roots and r be the common ratio. Then the roots become x_1, rx_1, r^2x_1 and $x_1(1 + r + r^2) = \frac{7}{2}$; $rx_1^2(1 + r + r^2) = -\frac{k}{2}$; $r^3x_1^3 = 1 \rightarrow rx_1\left(\frac{7}{2}\right) = -\frac{k}{2} \rightarrow rx_1 = -\frac{k}{7}$; $x_1 = -\frac{k}{7r}$
$r^3\left(-\frac{k}{7r}\right)^3 = 1$; $k = -7$ and $2x^3 - 7x^2 + 7x - 2 = 0 \rightarrow x = \frac{1}{2}, 1, 2$ and $r = 2$ or $\frac{1}{2}$. ***E***

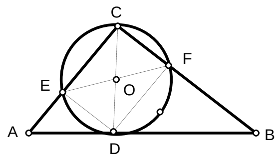
13. $-\cos\frac{\pi}{4} + \sin\frac{\pi}{4} + \tan 30^0 - \sec\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \sqrt{2} = \frac{\sqrt{3}-3\sqrt{2}}{3}$. ***D***

14. $R = (1000 - 20n)(6 + .25n)$ where n is the number of \$.15 increases. $n = 50$ and $n = -24$ and averaging $n = 13$
13(.25) = → new ticket price = \$9.25. **none of these E**

15. $R = (1000)(6) - (740)(9.25) = \$845$. ***A***

2009 Alpha Individual Test Solutions

16. $AC = 9$, $BC = 12$, $AB = 15$. $ABCD$ is a rectangle and $EF = CD$. A ΔABC = ½(108) = 54 = ½ $AB \cdot CD \rightarrow$ 108 = 15CD. $EF = CD = 7.2$ ***C***



17. $det \begin{bmatrix} 1 & x & 0 \\ 2 & 3 & -1 \\ -1 & 6 & x \end{bmatrix} = 0 \rightarrow 3x + 6 - x(2x - 1) = 0 \rightarrow 2x^2 - 4x - 6 = 0; x^2 - 2x - 3 = 0 \rightarrow -1, 3.$ ***C***

18. $\frac{4}{(n+3)(n+4)} = \frac{A}{(n+3)} + \frac{B}{(n+4)} \rightarrow A(n + 4) + B(n + 3) = 4; A + B = 0, 4A + 3B = 4 \rightarrow \sum_1^{30} \frac{4}{(n+3)(n+4)} = \frac{15}{17}$. ***B***

19. $(2cis\ 30^o)^8 = 2^8 cis\ 240^o; 256\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -128 - 128\sqrt{3}\, i.$ ***D***

20. $3x + 2y - z = 3$; $-2x - 2y - 2z = -4$; $4x - 3 = 5 \rightarrow x = 2$. ***D***
$\underline{x + y + z = 2}$ $\underline{2x + 3y + 2z = 3}$
$4x + 3y = 5$ $y = -1$

21. $\frac{sin\ x}{1 + cos\ x} + \frac{1 + cos\ x}{sin\ x} = \frac{sin^2 x + 1 + 2cosx + cos^2 x}{sinx\,(1+cosx)}; \frac{2(1+cosx)}{sinx\,(1+cosx)} = 2csc\,x.$ ***D***

22. $\frac{x-3}{2x+1} \geq 1; \frac{x-3}{2x+1} - 1 \geq 0 \rightarrow \frac{-x-4}{2x+1} \geq 0; \frac{x+4}{2x+1} \leq 0; x \neq -½; -4 \leq x < -½.$ [-4, - ½). ***C***

23. $sin(45 + 30)^o = \frac{30}{x} \rightarrow x = \frac{120}{\sqrt{6}+\sqrt{2}}; x = 30(\sqrt{6} - \sqrt{2}).$ ***A***

24. $P(x + 3) = a(x + 3)^2 + b(x + 3) + c = x^2 + 7x + 4 \rightarrow ax^2 + 6ax + 9a + bx + 3b + c \rightarrow a = 1, b = 1, c = -8.$ ***A***

25. $553_b + 670_b = 1003_b \rightarrow 5b^2 + 5b + 3 + 6b^2 + 7b = b^3 + 3; b^3 - 11b^2 - 12b = 0; b = 0, -1, 12.$ ***B***

26. $(a + bi)^2 = a^2 - b^2 - 2abi = -3 - 4i \rightarrow 2ab = -3, b = -\frac{2}{a}, b^4 - 3b^2 + 4 = 0, (b^2 - 4)(b^2 + 1) = 0, b = \pm 2, a = 1,$ (1, -2). ***C***

27. $f(f(x)) = \frac{1-f(x)}{1+f(x)} \rightarrow \frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}; \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x.$ ***A***

28. $\sqrt{3 + \sqrt{8}} - \sqrt{3 - \sqrt{8}} = x; x^2 = 3 + \sqrt{8} - 2\sqrt{9-8} + 3 - \sqrt{8}; x^2 = 4; x = 2,$ ***C***

29. $cot\ 2\theta - tan\ \theta = 1; \frac{1}{tan(2\theta)} - tan\ \theta = 1; \frac{1-tan^2\theta}{2tan\theta} - tan\theta = 1; 1 - 2tan^2\theta - 2tan^2\theta - 2tan\theta = 0$
$3tan^2\theta + 2tan\ \theta - 1 = 0; (3tan\ \theta - 1)(tan\ \theta + 1) = 0 \rightarrow tan\ \theta = 1/3, tan\ \theta = -1. \theta = tan^{-1}(1/3),$
$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$. ***D***

30. $4x^2 + 9y^2 - 4x - 3 = 0 \rightarrow \frac{(2x-1)^2}{4} + \frac{9y^2}{4} = 1 \rightarrow a = 2, b = 2/3, A = \frac{4}{3}\pi$ ***E***

Tie-Breakers:

1. $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1r_2} = \frac{9}{7}; \frac{1}{r_1r_2} = \frac{5}{7}; x^2 - \frac{9}{7}x + \frac{5}{7} = 0 \rightarrow 7x^2 - 9x + 5 = 0.$
2. (1423 + 1418)(1423 – 1418) = (2841)(5) = 14205
3. $9x + 9y\sqrt{2} + 7x\sqrt{2} + 14y = 1; 9x + 14y = 1; 9y + 7x = 0; \left(-\frac{9}{17}, \frac{7}{17}\right).$