Interschool Test Part B

Solutions

1. Since $sin18^O = \frac{-1+\sqrt{5}}{4}$. Since a and b are rational, $\frac{-1-\sqrt{5}}{4}$ is also a root. $r_1 + r_2 = -\frac{1}{2} and r_1 r_2 = -\frac{1}{4}$. $(a, b) \rightarrow (2, -1)$

2. Walk: m to a $\frac{12}{w}$ Total time: 5

row: m to b $\frac{12}{r-c}$ 7

row: $\frac{12}{r}$ $\frac{17}{3}$

Solving for $\frac{12}{w}$

$$\frac{12}{w} = 5 - \frac{12}{r+c} = 7 - \frac{12}{r-c} = \frac{17}{3} - \frac{12}{r}$$

$$\frac{12}{w} = \frac{5r+5c-12}{r+c} = \frac{7r-7c-12}{r-c} = \frac{17r-36}{3r}$$

$$\frac{12r-2c-24}{2r} = \frac{17r-36}{3r}; \ r = 3c$$

$$r = 9/2, w = 4, c = 3/2$$

3. For n squares on a side, the number of vertical walls is $(1 + 2+ 3+ \ldots + n) + n$. This is also the number of horizontal walls. Therefore the total number of walls, $T = \frac{2n(n+1)}{2} + 2n = n^2 + 3n$.
for $n = 16, 17$, $t = 304, 340$. Therefore the number of additional walls needed is 340 – 317 = 23 walls.

4. Red’s statements: “i did not kill don” and “shorty lied when he said i’m guilty,” are either both true or both false. Since it is given only one statement is false, they are both true. Therefore one of red’s other two statements are false.
shorty’s false statement was: “red is the guilty man.” Therefore his other three statements were true.
joey’s statement: “i never saw shorty before.” Contradicts one of shorty’s true statements and is therefore false. His other three statements were therefore true, including the fact he did not kill donavan and the fact that red was never in trenton.
since red told the truth about not having been in trenton, his false statement must have been that he never saw hank before.
tony’s last three statements have been shown to be true. Therefore it was false that hank lied about owning a revolver. Hank told the truth about not owning a revolver, about knowing red, and about being in philadelphia. Therefore he lied when he said: “I did not kill donavan.” thus hank is guilty.

5. Looking at special cases:

l	*W*	*I*	*S*	*A*	*S = 2w + 2l*	*I = (w - 1)*(l - 1)*
6	4	15	20	24	12 + 8 = 20	5*3 = 15
6	10	45	32	60	12 + 20 = 32	5*9 = 45
5	8	28	26	40	10 + 16 = 26	4*7 = 28
5	12	44	34	60	10 + 24 = 36	4*11 = 44

$$a = wl; \ \frac{S}{2} = w + l; \ I = wl - (w+l) + 1; \ I = A - \frac{S}{2} + 1 \rightarrow \ A = I + \frac{S}{2} - 1$$

6. Let *abc,abc,abc* represent any 9 digit number and $a(100{,}000{,}000) + b(10{,}000{,}000) + c(1{,}000{,}000) + a(100{,}000) + b(10{,}000) + c(1{,}000) + a(100) + b(10) + c$ represents the decimal representation of that number. Collecting like terms$\rightarrow a(100{,}100{,}100) + b(10{,}010{,}010) + c(1{,}001{,}001) \rightarrow$ $100a(1{,}001001) + 10b(1{,}001001) + c(1{,}001{,}001) = 1{,}001{,}001(100a + 10b + c)$.

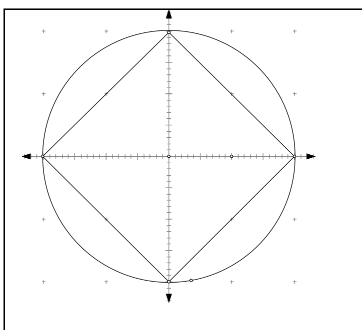
7. Let x be the amount of money tom had when he started.

Store 1	*Store 2*	Store 3	Store 4	Store 5
S $\frac{1}{2}x + 1$ (32) L $\frac{1}{2}x - 1$ (30)	*S* $\frac{1}{4}x + \frac{1}{2}$ (16) L $\frac{1}{4}x - \frac{3}{2}$ (14)	S $\frac{1}{8}x + \frac{1}{4}$ (8) L $\frac{1}{8}x - \frac{7}{4}$ (6)	S $\left(\frac{1}{16}x + \frac{1}{8}\right)$ L (4) $\frac{1}{16}x - \frac{15}{8}$ (2)	$\left(\frac{1}{32}x + \frac{1}{16}\right)$ (2) L 0

then: $x = \frac{1}{2}x + 1 + \frac{1}{4}x + \frac{1}{2} + \frac{1}{8}x + \frac{1}{4} + \frac{1}{16}x + \frac{1}{8} + \frac{1}{32}x + \frac{1}{16}$; $x = \frac{31}{32}x + \frac{62}{32} \rightarrow 32x = 31x + 62 \rightarrow x = \$62$

8. $19 - 6\sqrt{2} = (a + b\sqrt{2})^2 \rightarrow 19 - 6\sqrt{2} = a^2 + 2ab\sqrt{2} + 2b^2$; $19 = a^2 + 2b^2$ and $2ab = -6$; $b = -\frac{-3}{a} \rightarrow a^2 + 2\left(\frac{9}{a^2}\right) = 19$
$a^4 - 19a^2 + 18 = 0$; $(a^2 - 18)(a^2 - 1) = 0 \rightarrow a = \pm 1$ and $a = \pm 3\sqrt{2}$; **(1, -3)(-1, 3)**$(3\sqrt{2}, -\frac{\sqrt{2}}{2})(-3\sqrt{2}, -\frac{\sqrt{2}}{2})$

9. Find the area included between the following graphs: $x^2 + y^2 = 4$ and $|x|$ and $|y| = 2$.

	Area of sector = $\frac{1}{4}(4\pi)$ $= \pi$ Area of triangle = $\frac{1}{2}(4) = 2$ Area sector – triangle in 1st quadrant = $\pi - 2$ Area: $4\pi - 8$

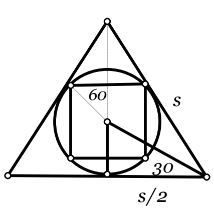
10. $|x| + |y| < 4$; integers means $x, y,$ must lie between (3, -3). In the first quadrant (3,0),(2,1),(2, 0) (1, 2),(1, 1))(1,0) 4 quadrants = 24. Add (0, 0) =25 ordered pairs.

$|x| + |y| < 11$; (10, 0)(9,1),(9,0)(8, 2)(8,1)(8,0)…(1.2)(1,1)(1,0) $\rightarrow 1 + 2 + 3 + \ldots + 10 = 55$; $55(4) + 1 = 221$.

$|x| + |y| < n$; $1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2} \rightarrow 4\cdot\frac{n(n-1)}{2} + 1 = 2n(n-1) + 1 = 2n^2 - 2n + 1$.

11. Area of triangle = $\frac{\sqrt{35(35-27)(35-25)(35-18)}}{2} \rightarrow \frac{\sqrt{35(8)(10)(17)}}{2} = \frac{20\sqrt{119}}{2} = 10\sqrt{119}$; $a\Delta = \frac{1}{2}h(27) = 10\sqrt{119} \rightarrow h = \frac{20\sqrt{119}}{27}$

12. $\sum_1^3 kx^{3-k} = 6 \rightarrow x^2 + 2x + 3 = 6$; $(x + 3)(x - 1) = 0$; $x = -3, 1$



13. $a = \frac{s\sqrt{3}}{6}$; $b = \frac{s\sqrt{3}}{3} \rightarrow a\Delta = \frac{1}{2}\cdot\frac{s}{2}\left(\frac{s\sqrt{3}}{3} + \frac{s\sqrt{3}}{6}\right) = \frac{s^2\sqrt{3}}{8}$; diagonal $= \frac{s\sqrt{3}}{3}\cdot\frac{1}{2}\cdot 2 = \frac{s\sqrt{3}}{3}$;
$a_{sq} = \frac{1}{2}\cdot\frac{s^2}{3} = \frac{s^2}{6}$ ratio $= \frac{s^2\sqrt{3}}{8} : \frac{s^2}{6} \rightarrow \frac{3\sqrt{3}}{4}$

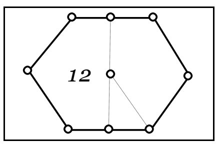
14. $N = 1, f(2) = {}^{5}/_{2}$; $n = 2.\ F(3) = 3$; $n = 3.\ F(4) = {}^{7}/_{2}$; $n = 4.\ F(5) = 4 \rightarrow$ $n = n.\ F(n+1) = \frac{n+3}{2}$ and $n = 2009$
$f(2009) = \frac{2012}{2} = 1006$

15. Let $u = x^2 + 4x + 6$ then $u^2 = u + 6$ and $(u - 3)(u + 2) = 0$ and $u = 3, -2 \rightarrow x^2 + 4x + 6 = 3$ and i: $x^2 + 4x + 3 = 0$; $x^2 + 4x + 6 = -2$ and ii: $x^2 + 4x + 8 = 0$. I has 2 real solutions and ii has no real solutions so -4 is the sum of the real solutions.

16. $\frac{3}{z-1} - \frac{1}{z} = \frac{3z-z+1}{z(z-1)} \rightarrow \frac{2\left(\frac{1-i\sqrt{3}}{2},\right)+1}{\left(\frac{1-i\sqrt{3}}{2}\right)^2 - \frac{1-i\sqrt{3}}{2}} = \frac{2-i\sqrt{3}}{\frac{-2-2\sqrt{3}}{4} - \frac{1-i\sqrt{3}}{2}} = \frac{2-i\sqrt{3}}{-1} \rightarrow -2 + i\sqrt{3}.$

17. $X^2 + 3 - x = 9$; $x^2 - x - 6 = 0 \rightarrow (x - 3)(x + 2) = 0$; $x = 3, -2$. Pts: $(3, 0), (-2, 5)$. $D = \sqrt{(3+2)^2 + 25} = 5\sqrt{2}$
$a + b = 7$

18.



$30 - 60 - 90 \rightarrow s/2 = \frac{6}{\sqrt{3}}$; $s = 4\sqrt{3}$, $p = 24\sqrt{3}$.

19. $Sin\ 2\theta = 2sin\theta cos\theta$. Let $(sin\theta + cos\theta) = x$, then $sin^2\theta + 2sin\theta cos\theta + cos^2\theta = x^2 \rightarrow 1 + a = x^2, x = \sqrt{a-1}$

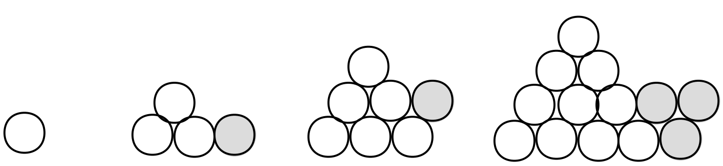
20. $log_{\sqrt{2}} 4 + log_4 \sqrt{2} - log_2 4 - log_4 2 = x$. $4 + \frac{1}{4} - 2 - \frac{1}{2} = \frac{7}{4}$

21. $1^2 + 2^2 + 3^2 + \dots + n^2 + \dots = \sum_1^k n^2$; $1\cdot2 + 2\cdot3 + 3\cdot4 + \dots + n(n+1) + = \sum_1^k n(n+1)$
$\sum_1^k n(n+1) - \sum_1^k n^2 = \sum_1^k n = 1 + 2 + 3 + \dots + k = 210 \rightarrow \frac{k(k+1)}{2} = 210$, $n^2 + n - 420 = 0$, $(k + 21)(k - 20) = 0$
$k = 20$

22. $A = ½$ and $(a + 1)(b + 1) = 2$; let $x = tan^{-1} a$ and $y = tan^{-1} b$ then $(tan\ x + 1)(tan\ y + 1) = 2$
$tan\ x\ tan\ y + tan\ x + tan\ y + 1 = 2 \rightarrow$ $tan\ x + tan\ y = 1 - tan\ x\ tan\ y$; $\frac{tan\ x + tan\ y}{1 - tan\ x\ tan} = 1$ and $tan\ (x + y) = 1$
then $x + y = \pi/4 = tan^{-1} a + tan^{-1} b$.

23. Let $a = \sqrt[3]{5 + 2\sqrt{13}}$ and $b = \sqrt[3]{5 - 2\sqrt{13}}$; $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b) \rightarrow$
$5 + 2\sqrt{13} + 5 - 2\sqrt{13} + 3\sqrt[3]{-27}x = x^3 \rightarrow x^3 + 9x - 10 = 0$; $(x - 1)(x^2 + x + 10)$. So $x = 1$.

24.



0 *1* *1* *3*

N	m
1	0
2	1
3	1
4	3
5	3
6	6
7	6
8	10
9	10

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Separating into 2 tables for odd and even number of rows.

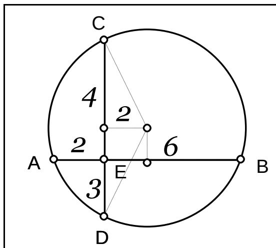
N	m	N	m
1	0	2	1
3	1	4	3
5	3	6	6
7	6	8	10
9	10		
Difference is of the form of n^2 $(n^2 - 1)/8$ if n is odd.		Difference is of the form of n^2 $N(n + 2)/8$ if N is even.	

25. There are eight vertices and a triangle has 3 vertices so: ${}_8C_3 = \frac{8!}{3!5!} = 56$

26. A right triangle with hypotenuse of 10 cm can be inscribed in a semi-circle. The largest area will br a 45-45-90 degree triangle. $A = ½\left(\frac{10}{\sqrt{2}} \cdot \frac{10}{\sqrt{2}}\right) = 25\text{ cm}^2$.

27. Let $a = (x - 3)$, $b = (x + 4)$ and $c = a + b$. Then $a^3 + b^3 = (a + b)^3 = a^3 + b^3 + 3ab(a + b)$. Then $3ab = 0$ and $a + b = 0$. Substituting: $x - 3 + x + 4 = 0$ and $x = -½$ and $x = 3$ and $x = -4$. The smallest root is -4.

28.



$R^2 = 2^2 + (7/2)^2$

$= 4 + \frac{49}{4}$

$= \frac{65}{4}$

$R = \frac{\sqrt{65}}{2}$

$D = \sqrt{65}$

29. $61_b = 6b + 1$; $51_b + 5b + 1$; $3731_b = b^3 + 3b^2 + 7b + 3$; $(6b + 1)(5b + 1) = 3b^3 + 7b^2 + 3b + 1$
$30b^2 + 11b + 1 = b^3 + 3b^2 + 7b + 3 \rightarrow 3b^3 - 23b^2 - 8b = 0 \rightarrow b(3b + 1)(b - 8) = 0 \rightarrow b = 8$.

30. $F(0) = 2, f(1) = -2; f(n + 1) = f(n) - f(n - 1)$ for $n \geq 1$. Find $f(2009)$.

n	$f(n + 1)$	n	$f(n + 1)$	
0	2			Since the function is cyclic and it repeats After 6 iterations: 2009/6 has a remainder of 5 and f(2009) would be 4,
1	- 2	6	2	
2	- 4	7	-2	
3	- 2	8	-4	
4	2	9	-2	
5	4	10	2	

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31. $\frac{\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}}{\sqrt{6+4\sqrt{2}}} = \frac{1+\sqrt{2}-(\sqrt{2}-1)}{\sqrt{2}\,(1+\sqrt{2})} = \frac{2}{2+\sqrt{2}} = \frac{2(2-\sqrt{2})}{2} = 2-\sqrt{2} \rightarrow (a, b) = (2, 2).$

32. $A + b + c = 11$. $100a + 10b + c - 99 = 100c + 10b + a$, $99a - 99c = 99 \rightarrow a - c = 1$, $3a + 3b = 8c$.
solving the three equations: $a = 2$, $b = 6$, $c = 3$ and $abc = 36$.

33. $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{10}{3} \rightarrow 3\sin^2\theta + 3\cos^2\theta = 10\sin\theta\cos\theta \rightarrow (3\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0 \rightarrow \cot\theta = 3$, $\tan\theta + \frac{1}{3}$;
$\mathbf{\sec\theta = \frac{\sqrt{10}}{3}}$**.**

34. $\frac{c}{r-5} = 2$; $\frac{r-5}{c-25} = 3$. Where c = the number of original crows and r = the original number of robins.
$c - 2r = -5$ and $3c - r = 20 \rightarrow r = 20$.

35. $(\sqrt{75} - \sqrt{12})^2 = 27 \rightarrow \sqrt[3]{\frac{1}{27}} = \mathbf{\frac{1}{3}}$

36. For 8 squares to form a single square on the addition of one man, the original squares must have been 35 on a side and included a total of 9800 men. When harold joined them they would have formed a single square of 9801, 99 on a side. The soldiers were armed with pikes and axes, facing a norman army of the same size with long bows and cavalry. Harold's saxon soldiers were slaughtered. With the norman conquest, england shifted from a tribal to a feudal society and the high middle ages began. (noggin twisters.com)

37. 15 + 10 + 17; 10 + 14 + 18; 15 + 14 + 13; $x + y + 15 = 42$; $y + 31 = 42 \rightarrow 11$; $x = 16$; $30 + z = 42 \rightarrow z = 12$; $x + y + z = 39$

15	10	**17**
X= 16	14	*Z*= 12
Y=11	**18**	**13**

38. $lim_{x\rightarrow 27} \frac{\sqrt{1+\sqrt[3]{x}}-2}{x-27} = lim_{x\rightarrow 27} \frac{0}{0} \rightarrow$ using h'osptal rule gives $lim_{x\rightarrow 27} \frac{\left(1+x^{1/3}\right)^{1/2}-2}{x-27}$;
$lim_{x\rightarrow 27} \frac{\frac{1}{2}\cdot\frac{1}{3}x^{-2/3}(1+x^{1/3})^{-1/2}}{1} = \frac{1}{108}$

39. $2\sqrt{(x-3)^2 + (y-6)^2} = \sqrt{(x+2)^2 + (y-5)^2}$; $4(x^2 - 6x + y^2 - 12y + 45) = x^2 + 4x + y^2 - 10y + 29 = 0$
$3x^2 - 28x + 3y^2 - 38y + 151 = 0$

40. $\frac{3}{11} = 0.230769230\ldots$ there is a 6 digit repeat so 2009/6 has a remainder of 5 so the 2009th digit is 6.

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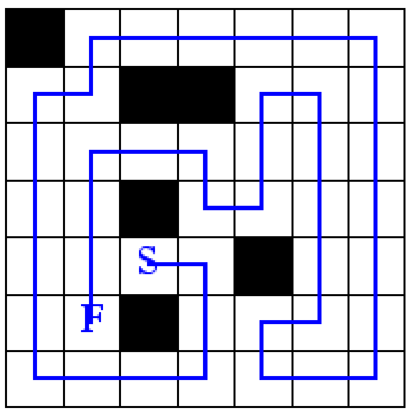
41. After a win, amount at the time of the bet a_i becomes $\frac{3}{2}A_i$ and after a win, amount at the time of the bet a_j becomes $\frac{1}{2}A_j$. After three wins and three loses in any order is: $\left(\frac{3}{2}A_{i_1}\right)\left(\frac{3}{2}A_{i_2}\right)\left(\frac{3}{2}A_{i_3}\right)\left(\frac{1}{2}A_{j1}\right)\left(\frac{1}{2}A_{j2}\right)\left(\frac{1}{2}A_{j3}\right)$ which equals $\frac{27}{64}A$ where a is the original amount. So $64 - $27 = $37 loss.

42. $log_{\frac{1}{125}} 25\sqrt[3]{625} = x$; $5^{-3x} = 5^2 \cdot 5^{\frac{4}{3}}$, then $-3x = \frac{10}{3}$ and $x = -\frac{10}{9}$.

43.



44. Doug and Joe are married, with Ken being the bachelor. Ken and Joe have blue eyes and Doug has brown eyes and is bearded.

45. One solution is to label the coins with the letters fake mind clot and weigh the coins in the following three combinations:

ma do – like   me to – find   fake -- coin

logic will now allow you to find the fake coin based on the three results. For instance, if the results were left down, balanced, left down, we could work out which coin is fake in the following way:
from the middle weighing we know that the coins metofind are all normal. So one of the coins ackl is fake. Therefore looking at these coins one at a time in the other two weighings, we can see that:

a - appears on the left twice and could be fake.
C - appears only once, therefore can't be fake (otherwise the first weighing would be balanced).
K - appears on opposite sides, so it can't make the left side go down both times.
L - appears only once, therefore can't be fake (otherwise the third weighing would be balanced).

Therefore the only possibility is a, which must be heavier. Any other combination of ups and downs will allow you to use the same logic to find the fake coin.