Vectors

Mu Alpha Theta National Convention
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General Instructions:

1. Unless otherwise stated all answers should be written as decimals.
2. If you are asked to give your answer as a fraction, please give your answer in \( \frac{a}{b} \) form where \( a \) and \( b \) are relatively prime.
3. Boldface notation indicates vectors.

Questions

1. What is the product of the values of \( a \) such that \( \mathbf{A} = ai - 2j + k \) and \( \mathbf{B} = 2ai + aj - 4k \) are perpendicular?

2. Find the acute angle, in degrees, between the vectors \( \mathbf{A} = 2i + 2j - k \) and \( \mathbf{B} = 6i - 3j + 2k \). Give your answer to the nearest degree.

3. Simplify \( 2\mathbf{A} + \mathbf{B} + 3\mathbf{C} - \left( \mathbf{A} - 2\mathbf{B} - 2 \left( 2\mathbf{A} - 3\mathbf{B} - \mathbf{C} \right) \right) \). Write your answer as the sum of the coefficients of \( \mathbf{A} \), \( \mathbf{B} \) and \( \mathbf{C} \).

4. Evaluate \( k \cdot (i + j) \).

5. Two vectors of magnitude 6 are orthogonal. What is the magnitude of the resultant sum of these two vectors?

6. Let \( \mathbf{v}_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \) and \( \mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \). Find the acute angle, in degrees, between \( \mathbf{v}_3 \) and \( \mathbf{v}_1 \). Note: an alternate notation for \( \mathbf{v}_1 \) is \( 3i + 5j \) and \( \mathbf{v}_2 \) is \( i + 7j \). Give answer rounded to four significant digits.

7. Evaluate \( (2i - 3j) \cdot \left[ (i + j - k) \times (3i - k) \right] \).

8. An automobile travels 3 miles due north, then 5 miles northeast. What is the magnitude, in miles, of the resultant vector? Give your answer rounded to four significant digits.

9. Determine the value of \( a \) so that \( \mathbf{A} = 2i + aj + k \) and \( \mathbf{B} = 4i - 2j - 2k \) are perpendicular.

10. Find the largest of the three acute angles, to the nearest degree, which the line joining the points \( (1, -3, 2) \) and \( (3, -5, 1) \) makes with each of the coordinate axes.
11. Find the area of a parallelogram having diagonals defined by \( \mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} \) and \( \mathbf{B} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \).

12. Two boats are at the same place at 5:00 a.m. on a certain day. Starting at 5:00 a.m., the first boat is sailed 7.23° to the west of north at a rate of 11.44 miles per hour. The second boat does not move until 7:00 a.m. on the same day at which time it is sailed at 12.38 miles per hour in a direction 10.25° to the east of north. What is the distance, in miles, between the boats at 11:30 a.m. on the same day?

13. Find the sum of all values of \( m \) such that the vector \( 4\mathbf{i} + m\mathbf{j} + \mathbf{k} \) is perpendicular to the vector \( m\mathbf{i} + (2m + 1)\mathbf{j} - 3\mathbf{k} \). Give your answer as a simplified fraction in lowest terms.

14. The two planes \( 2x + 3y - 4z = 12 \) and \( x - y + z = -3 \) intersect in line \( L \). The line whose equation is \( \frac{x + 2}{-2} = \frac{y - 1}{z - 4} = \frac{z}{7} \) and line \( L \) intersect at point \( (a, b, c) \). Find \( a + b + c \).

15. Two sides of a triangle are formed by the vectors \( \mathbf{A} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} \) and \( \mathbf{B} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k} \). Determine the smallest angle to the nearest degree.

16. A plane contains the points \((1, 2, -3)\), \((0, 4, 6)\) and \((2, 3, -4)\). Find a vector perpendicular to the plane in the form \( a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \) where \( a, b \) and \( c \) are relatively prime integers with \( a > 0 \). Give your answer as the value of \( a + b + c \).

17. Find the shortest distance from \((6, -4, 4)\) to the line joining \((2, 1, 2)\) and \((3, -1, 4)\).

18. Find the length of the projection of the vector \( 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} \) on the vector \( \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \). Write your answer as a simplified fraction in lowest terms.

19. Two radio stations receive a distress message from an island. Station A is 120 miles due north of Station B. Station A receives the message at a bearing of \( S 50^\circ \text{E} \) while Station B receives it at a bearing of \( N 47^\circ \text{E} \). How many minutes will a helicopter flying at 110 mph take to reach the island from Station A? Give your answer rounded to four significant digits. [Note: \( S 50^\circ \text{E} \) means 50° east of due south.]

20. Find the volume of the parallelepiped whose edges are represented by the vectors:

\[
\begin{align*}
\mathbf{A} &= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \\
\mathbf{B} &= \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\mathbf{C} &= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}
\end{align*}
\]