Mu Alpha Theta National Convention: Denver, 2001
Gemini Test – Alpha Division

1. It takes Amanda, James, and Lindsay 4 hours to paint a certain fence, working together. It takes Amanda, James, and Ravi 5 hours to paint the same fence. It takes Amanda, Lindsay, and Ravi 6 hours to paint the fence, but it takes James, Lindsay, and Ravi 7 hours to paint the fence. How many hours would it take Amanda, James, Lindsay, and Ravi, working together, to paint the fence?

   (A) 3 (B) $\frac{1260}{319}$ (C) $\frac{63}{16}$ (D) $\frac{24}{7}$ (E) NOTA

2. An equilateral triangle has an area of 1. One side of the triangle also serves as the side of a square. Find the area of that square.

   (A) $\sqrt{3}$ (B) $\frac{4\sqrt{3}}{3}$ (C) $\frac{2\sqrt{6}}{3}$ (D) $4-\sqrt{3}$ (E) NOTA

3. Find the absolute value of the difference between the non-real roots of the equation $x^3+4x^2+x-26=0$.

   (A) 2 (B) 4 (C) 6 (D) 13 (E) NOTA

4. Find the probability that three different cards selected at random from a standard deck of 52 playing cards will all be of the same suit.

   (A) $\frac{1}{16}$ (B) $\frac{22}{425}$ (C) $\frac{24}{425}$ (D) $\frac{16}{289}$ (E) NOTA

5. Evaluate: $\sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{10}\right)+\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{2\pi}{5}\right)$

   (A) 0 (B) 1 (C) $\frac{7}{4}$ (D) 2 (E) NOTA

6. Solve for x: $\begin{vmatrix} x+12 & 2x+22 & 1 \\ x & 2x+2 & 0 \\ 12 & 22 & 1 \end{vmatrix}=12$.

   (A) -6 (B) -2 (C) 2 (D) 6 (E) NOTA

7. Find the area of the circle which passes through the points (1, -2), (-4, 3), and (7, 14).

   (A) 73π (B) 121π (C) 144π (D) 148π (E) NOTA

8. 185 whales walk into a bar. The bartender asks, "What'll it be?" The whales explain that only 3/5s of them plan to drink and the rest are designated swimmers. Of those drinking, 12 plan to drink only saltwater, 8 are going to drink only fresh water, x whales plan to drink only carbonated water, $5x$ are going to drink only ice water, and one lonely whale plans to just drink tap water. What is x?

(A) 12 (B) 13 (C) 14 (D) 15 (E) NOTA

9. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect perpendicularly at E. If $AB = 10$, $CD = 14$, and $AE = 6$, what is the area of O?

(A) 49π (B) 50π (C) 51π (D) 52π (E) NOTA

10. Find the area of the triangle whose vertices lie on the points (1, 2, 3), (4, 6, 15), and (6, 14, 87)?

(A) 85 (B) 102 (C) 128 (D) 136 (E) NOTA

11. Express the square of 111111_7 in base seven.

(A) 11111111111_7 (B) 22222222222_7
(C) 13612230531_7 (D) 25010211642_7 (E) NOTA

12. The Lecroix Jazz club has a total of 150 seats divided into three sections. The two smaller sections have an equal number of seats. The larger section has a number of seats which is an integer multiple of the number of seats in one of the smaller sections. What is the smallest number of seats that the larger section could contain?

(A) 75 (B) 90 (C) 100 (D) 120 (E) NOTA

13. What is the sum of the digits in the binary representation of the base-ten number 2001?

(A) 4 (B) 5 (C) 6 (D) 7 (E) NOTA

14. Find the area of a quadrilateral with consecutive sides of lengths $\sqrt{13}$, 5, $\sqrt{41}$, and $\sqrt{29}$ whose diagonals are perpendicular and have integer lengths.

(A) 24 (B) 29 (C) 41 (D) 54 (E) NOTA

15. What is the range of the areas of all non-degenerate hexagons with perimeter 36?

(A) $[0,\ 36\sqrt{3}]$ (B) $[0,\ 54\sqrt{3}]$ (C) $(0,\ 54\sqrt{3}]$ (D) $(0,\ 36\sqrt{3}]$ (E) NOTA

16. If $f(x) = ax + b$ and $g(x) = cx + d$, which of the following is a relationship between *a*, *b*, *c*, and *d* that will guarantee that $f(g(x)) = g(f(x))$?

(A) $ac = 1$ (B) $\frac{c-1}{a-1} = \frac{d}{b}$ (C) $ad + b = 0$ (D) $b + ad = 1 - ac$ (E) NOTA

17. Evaluate: $\sum_{n=1}^{20}((n+1)(n+2))$

(A) 3540 (B) 3580 (C) 3600 (D) 6410 (E) NOTA

18. In triangle *ABC*, $AB = 5$, $AC = 7$, and the measure of angle *BAC* is 19°. Determine the measure of angle *ABC* to the nearest tenth of a degree.

(A) 125.4° (B) 122.7° (C) 119.2° (D) 54.6° (E) NOTA

19. Determine the sum of all values of *x* for which $\log_3 x^4 + \log_x 3^{21} = 20$.

(A) 5 (B) $\frac{15}{2}$ (C) 10 (D) $30\sqrt{3}$ (E) NOTA

20. Find the product of the positive proper integral factors of 108.

(A) 108^2 (B) 108^3 (C) 108^4 (D) 108^5 (E) NOTA

21. Evaluate: $\sqrt{\frac{15}{2} + \frac{1}{2}\sqrt{\frac{15}{2} + \frac{1}{2}\sqrt{\frac{15}{2} + \cdots}}}$

(A) $\sqrt{6}$ (B) 3 (C) $\sqrt{15}$ (D) $1 + \sqrt{5}$ (E) NOTA

22. Three unit circles lie in a plane such that the centers of the circles form the vertices of an equilateral triangle with sides of length 2. What is the area of the portion of the triangle that lies outside all three of the circles?

(A) $\pi - \sqrt{3}$ (B) $\sqrt{3} - \frac{\pi}{2}$ (C) $\frac{\sqrt{3}}{3} + \frac{\pi}{2}$ (D) $\frac{1}{6}$ (E) NOTA

23. Given the set of the first twenty Fibonacci numbers, $\{1,1,2,3,5,\ldots\}$, list the mean, mode, and median of the set in order of increasing magnitude.

(A) mean, median, mode (B) mode, mean, median
(C) mode, median, mean (D) median, mean, mode (E) NOTA

24. Find the area of the smallest ellipse that can contain two non-overlapping equilateral triangles each with side lengths of 2.

(A) $2\sqrt{3}+\pi$ (B) $\frac{\pi\sqrt{3}}{2}$ (C) $\pi\sqrt{3}$ (D) 2π (E) NOTA

25. An equilateral triangle has sides of length $2n$, where n is a natural number. What is the greatest number of unit circles whose centers could be placed on or in the triangle with no two circles overlapping? (The circles may be tangent.)

(A) $\frac{n(n+1)}{2}$ (B) $\frac{(n+1)(n+2)}{2}$
(C) $\frac{n(n+1)(2n+1)}{6}$ (D) n^2 (E) NOTA

26. Let N be a natural number and a perfect square. If D is the number of positive proper integral divisors of N, then what is the remainder when 12^D is divided by 13?

(A) 1 (B) 3 (C) 10 (D) 12 (E) NOTA

27. Evaluate: $\sum_{n=0}^{\infty}\left(n\left(\frac{2}{3}\right)^n\right)$

(A) 3 (B) 6 (C) 9 (D) 12 (E) NOTA

28. Bill was in such a hurry to eat the brownies he'd baked that he made 8 sloppy cuts with his knife, producing brownies of various shapes and sizes. What is the greatest number of brownies he could have cut? Assume that the uncut brownie was a right rectangular prism and that all cuts made were perpendicular to the face with the largest area.

(A) 37 (B) 46 (C) 56 (D) 60 (E) NOTA

29. Simplify: $\sqrt{\dfrac{3}{\sqrt[3]{4}-1}}$

(A) $\dfrac{3\sqrt{2}}{2}$ (B) $\sqrt[3]{2}+1$ (C) $\sqrt{5}$ (D) $2\sqrt[3]{4}-1$ (E) NOTA

30. For positive numbers *a*, *b*, and *c*, which of the following is never greater than $\dfrac{a+b+c}{2}$?

(A) $\dfrac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}$ (B) $\dfrac{a^2+b^2+c^2}{2}$
(C) $\dfrac{3\sqrt{abc}}{2}$ (D) $a-b-c$ (E) NOTA

31. Given that *x*, *y*, and *z* are real, solve for *x*:

$$xy^4+xy^2z^2-5y^4-5y^2z^2+3xy^2+xz^2-15y^2-5z^2+2x=10$$

(A) 1 (B) 2
(C) 5 (D) cannot be determined (E) NOTA

32. Let F_n represent the *n*th Fibonacci number. Simplify: $\dfrac{(F_{n+1})^2-(F_n)^2}{F_{n+2}}$

(A) F_{n-1} (B) F_n-2 (C) $2F_n$ (D) $4F_{n-1}$ (E) NOTA

33. A triangle with sides 6, 8, and 10 is rotated about one of its sides to produce a solid. What is the difference between the greatest and least possible surface areas of such a solid?

(A) $\dfrac{144\pi}{5}$ (B) 48π (C) $\dfrac{96\pi}{5}$ (D) $\dfrac{384\pi}{5}$ (E) NOTA

34. Determine the sum of all values of *x*, $0\le x<2\pi$, for which $\cos(4x)=3\sin(2x)+2$.

(A) 10π (B) $\dfrac{21\pi}{2}$ (C) 14π (D) 15π (E) NOTA

35. A triangle lies on the xy-plane with vertices at points (a, 2), (3, b), and (1, 1). If the point (a, b) lies on the graph (x-1)(y-1) = 12, what is the area of the triangle?

(A) 5 (B) 6 (C) 10 (D) 12 (E) NOTA

36. Farmers know that in a flock of crows, each pair of crows either mutually trusts or mutually distrusts each other. Knowing nothing of the individual attitudes of the crows, a farmer looks at a flock, counts them, and tells his son, "There is a group of three of the crows in the flock who all trust one another or else there is a group of three of the crows in that flock who all mistrust one another." What is the smallest possible number of crows in the flock for the farmer to have known what he told to his son?

(A) 5 (B) 6 (C) 7 (D) 8 (E) NOTA

37. Roman Turek writes his name on a slip of paper and cuts the paper into ten pieces, each containing exactly one of the letters in his name. He then tosses each of ten pieces into one of three nets, choosing the net for each piece independently and uniformly at random. He observes that each net has at least one letter. What is the probability that the 'k' and the 'o' are in the same net?

(A) $\frac{1}{3}$ (B) $\frac{2}{9}$ (C) $\frac{7}{27}$ (D) $\frac{5}{27}$ (E) NOTA

38. Bill McLean plays a game with a regular fair 6-sided die. First, he makes one roll to establish a reference number, the "point" number. He then continues to roll until he again rolls the "point" established on the first roll and quits rolling. He then adds together the numbers from each roll (including the first and last rolls of the "point") to get the sum, S, of all the rolls. What is the expected value of S?

(A) 18 (B) 21 (C) $\frac{49}{2}$ (D) 28 (E) NOTA

39. If Nozick and Rawls each begin flipping a fair coin and each continues until he has flipped heads exactly twice, what is the probability that they will both stop after the same number of flips?

(A) $\frac{1}{2}$ (B) $\frac{2}{9}$ (C) $\frac{4}{27}$ (D) $\frac{5}{27}$ (E) NOTA

40. What is the sum of the distinct complex numbers that satisfy the equation $x^5 - 26x^4 + 268x^3 - 1368x^2 + 3456x - 3456 = 0$?

(A) 4 (B) 6 (C) 10 (D) 26 (E) NOTA