Mu Alpha Theta National Convention: Denver, 2001
Gemini Test – Mu Division

1. Determine the sum of all real values of x for which $x^4 + 4x^3 + 6x^2 + 4x = 15$.

   (A) 1 (B) –3 (C) –4 (D) -2 (E) NOTA

2. Solve for a: $\lim_{x \to 2a} x^2 = \frac{a}{2}$

   (A) 0 and $\frac{1}{8}$ (B) 0 and $\frac{1}{2}$ (C) 0 and $\frac{1}{4}$ (D) 0 and $\frac{1}{16}$ (E) NOTA

3. An airplane travels 2050 miles in the same time a car travels 260 miles. If the speed of the plane is 358 miles per hour greater than the speed of the car, find the airplane's speed.

   (A) 314 mph (B) 410 mph (C) 434 mph (D) 300 mph (E) NOTA

4. Find the 405^{247}th derivative of $f(x) = \sin x$.

   (A) $\sin x$ (B) $-\cos x$ (C) $-\sin x$ (D) $\cos x$ (E) NOTA

5. If $2^{x^2+2x} = 7$, what is $(x+1)^2$?

   (A) 14 (B) $2\log_2 3$ (C) $\log_2 7 + 1$ (D) 9 (E) NOTA

6. Evaluate: $\int_0^1 \sin(2000\pi x)\sin(2001\pi x)\ dx$

   (A) $\frac{1}{2}$ (B) $\frac{1}{2\pi}$ (C) $\frac{\pi}{4}$ (D) 0 (E) NOTA

7. Urn I contains 3x black and 2y red marbles while Urn II has 2x red and 5y black marbles. Richard randomly takes an urn and draws a marble from it. Given that he picked a black marble, what's the probability it came from Urn II? Note: x and y are positive integers.

   (A) $\frac{10y^2 + 15xy}{6x^2 + 30xy + 10y^2}$ (B) $\frac{3x + 5y}{5x + 7y}$

   (C) $\frac{5y}{2x + 5y}$ (D) $\frac{15xy}{6x^2 + 19xy + 10y^2}$ (E) NOTA

8. Find the total area bounded by the function $f(x)$ and the x-axis if $f(x)$ is piecewise defined by

$$f(x)=\begin{cases} 2(x+1)(x+3) & -3\le x\le -1 \\ x-x^3 & -1\le x\le 1 \\ -2x^2+8x-6 & 1\le x\le 3 \end{cases}.$$

Note that area is always treated as a positive number.

(A) $\frac{35}{6}$ (B) $\frac{16}{3}$ (C) 4 (D) $\frac{8}{3}$ (E) NOTA

9. Given that $\sin x=\frac{\sqrt{7}}{6}$, evaluate $\cos(2x)$.

(A) $\frac{11}{18}$ (B) $\frac{4}{9}$ (C) $\frac{\sqrt{203}}{18}$ (D) $\frac{5}{36}$ (E) NOTA

10. Suppose b and d are positive, prime integers. How many values of b are there such that $P(x)=16x^3+bx^2+3x+d$ has an inverse function for all real x?

(A) 4 (B) 5 (C) 3 (D) 2 (E) NOTA

11. A triangle has vertices at (–1, 6, 4), (2, 8, –7), and (3, 0, 9). Where do its medians intersect?

(A) $\left(2,4,\frac{7}{3}\right)$ (B) $\left(\frac{11}{3},\frac{7}{3},\frac{8}{3}\right)$ (C) $\left(\frac{4}{3},\frac{14}{3},2\right)$ (D) $\left(-2,0,\frac{4}{3}\right)$ (E) NOTA

12. Two functions $f(x)$ and $g(x)$ are called *friendly* if $f'(x)=g(x)$ and $g'(x)=f(x)$. Find $\lfloor 100(f(1)+g(1))\rfloor$ given that $f(0)=0$ and $g(0)=1$. Note: $\lfloor x\rfloor$ represents the greatest integer less than x.

(A) 314 (B) 135 (C) 271 (D) 738 (E) NOTA

13. Compute $\binom{n}{5}$, where n satisfies the equation $56=\binom{n-2}{5}+2\binom{n-2}{4}+\binom{n-2}{3}$.

(A) 6 (B) 28 (C) 84 (D) 56 (E) NOTA

14. Due to a manufacturing glitch, one out of every *n* Pokémon game cartridges Nintendo produces is faulty. In order to save money on testing costs, quality control takes *n* boxes with *n* cartridges each and randomly tests one from each box. As *n* increases, the probability quality control does <u>*not*</u> test a defective cartridge approaches *p*. Evaluate $\ln p$.

(A) –1 (B) –2 (C) –3 (D) –0.5 (E) NOTA

15. At Apu's convenient store, a bag of chips, one soda, and two candy bars cost $5.00. At the same store, two bags of chips, four sodas, and two candy bars cost $12.00. Finally, two bags of chips, seven sodas, and five candy bars cost $18.00. How much do five sodas and three candy bars cost?

(A) $9.00 (B) $10.00 (C) $11.00 (D) $12.00 (E) NOTA

16. Define a sequence so that $a_1 = \sin\theta$, $a_{n+1} = \cos a_n$ for odd *n*, and $a_{n+1} = \sin a_n$ for even *n*.

Evaluate $\frac{d}{d\theta}a_{100}$ at $\theta = \frac{\pi}{2}$.

(A) 1 (B) 0 (C) sin 1 (D) –1 (E) NOTA

17. How many ways can *r* people from a group of *n* be seated in a round table?

(A) $\frac{n!(r-1)!}{(n-r)!}$ (B) $\frac{n!}{(n-r)!r}$ (C) $\frac{n!}{(n-r)!}$ (D) $\frac{n!}{(r-1)!}$ (E) NOTA

18. In order to vary his exercise routine, Richard decides that everyday, he will jog *x* + *z* miles at *y* miles per hour and afterwards, ride a bike for *y* miles at *x* + *z* miles per hour, where *x*, *y*, and *z* are random positive numbers. What's the fastest time Richard can finish this type of routine?

(A) 2 hours (B) $\frac{1}{16}$ day (C) 10800 sec (D) 60 min (E) NOTA

19. Let *a*, *p*, and *b* be distinct natural numbers less than or equal to 50, with *p* a prime. Find the maximum value of $\frac{p}{2}\left(\left\lfloor \frac{b}{p} \right\rfloor^2 - \left\lfloor \frac{a}{p} \right\rfloor^2 + \left\lfloor \frac{a}{p} \right\rfloor + \left\lfloor \frac{b}{p} \right\rfloor\right)$. Note: $\lfloor x \rfloor$ represents the greatest integer less than *x*.

(A) 675 (B) 475 (C) 875 (D) 1275 (E) NOTA

20. Given the recursive progression $a_{n+1} = \dfrac{a_n^p\left(p-1\right)+c}{pa_n^{p-1}}$ for p, $c > 0$, find $\lim\limits_{n\to\infty} a_n$, given that such a limit exists.

(A) c^2 (B) c^p (C) $c^{\frac{1}{p}}$ (D) $\sqrt{c}$ (E) NOTA

21. A set of n lattice points is randomly chosen in the three-dimensional Cartesian space. What is the minimum value of n that will guarantee that there is at least one pair of lattice points in the set such that the midpoint of the segment connecting the two points is also a lattice point? Note: a lattice point is a point with integer coordinates.

(A) 7 (B) 5 (C) 8 (D) 9 (E) NOTA

22. Determine the coefficient of $x^2y^2z^2$ in the simplified expansion of $(1+x)^5(1+y)^4(1+z)^3$.

(A) 1 (B) 20 (C) 80 (D) 180 (E) NOTA

23. If $\lim\limits_{x\to a}(f(x)+g(x))=2$ and $\lim\limits_{x\to a}(f(x)-g(x))=1$, determine $\lim\limits_{x\to a}\left(f(x)g(x)\right)$.

(A) 3 (B) $\dfrac{3}{4}$
(C) diverges (D) cannot be determined (E) NOTA

24. A fifth-degree polynomial $P(x)$ with a leading coefficient of one has roots whose sum is 207. Find the coefficient of the x^5 th term of $xP(x+3)+3P(x+3)$.

(A) –189 (B) –195 (C) –201 (D) –207 (E) NOTA

25. Two posts, one 30 feet high and the other 40 feet high, stand 240 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. To the nearest integer, how far from the taller post should the stake be placed to use the least amount of wire? There is no slack on the wires and the posts are perpendicular to the ground.

(A) 142 ft (B) 103 ft (C) 30 ft (D) 137 ft (E) NOTA

26. In terms of R and H, what's the maximum volume of a cylinder that can be inscribed in a right circular cone of radius R and height H?

(A) $\dfrac{4\pi HR^2}{27}$ (B) $\dfrac{\pi HR^2}{3}$ (C) $\dfrac{2\pi HR^2}{9}$ (D) $\dfrac{\pi HR^2}{8}$ (E) NOTA

27. A sequence is defined recursively as: $a_0 = 3,\ a_1 = 1,\ a_n = \dfrac{3a_{n-2} - a_{n-1}}{2}$ for $n \ge 2$. Evaluate $\lim\limits_{n\to\infty}(a_n)$.

(A) $\dfrac{11}{5}$ (B) 1 (C) $-\dfrac{11}{5}$ (D) diverges (E) NOTA

28. Find $\dfrac{d}{dx} f^{-1}(x)$ at $x = 0$ if $f(x) = \int_2^x \dfrac{dt}{\sqrt{1+t^7}}$.

(A) Undefined (B) $-\sqrt[7]{\dfrac{3}{4}}$ (C) 1 (D) $\sqrt{129}$ (E) NOTA

29. How many ways can a loving mother distribute 25 identical candy bars to her nine children so that each child gets at least one?

(A) 735467 (B) 735468 (C) 735769 (D) 735470 (E) NOTA

30. A coin is flipped. If the coin shows heads, two dice are rolled; if the coin shows tails, three dice are rolled. If the result of this process is that the sum of the numbers on the dice is eight, what is the probability that the coin showed heads?

(A) $\frac{10}{17}$ (B) $\frac{7}{10}$ (C) $\frac{223}{432}$ (D) $\frac{1}{2}$ (E) NOTA

31. Evaluate: $\cos(\text{Arctan}(\sin(\text{Arccot}(x))))$

(A) $\sqrt{\frac{x}{x+1}}$ (B) $\sqrt{\frac{x}{x^2+1}}$ (C) $\sqrt{\frac{x^2}{x^2+1}}$ (D) $\sqrt{\frac{x^2+1}{x^2+2}}$ (E) NOTA

32. If $f(x) = -2xe^x \cos x$, evaluate $f'(0)$.

(A) -1 (B) 0 (C) 1 (D) 2 (E) NOTA

33. The sum of the series $\sum_{x=1}^{1000} \frac{1}{x^2+3x+2}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Evaluate $m+n$.

(A) 2863 (B) 751 (C) 107 (D) 3643 (E) NOTA

34. A stick of unit length is randomly broken into two pieces at a point. On average, what is the ratio of the shorter length to the longer one?

(A) $\left(\ln\frac{81}{16}-1\right):1$ (B) 1:2
(C) $(\ln 256 - 4):3$ (D) $(2\ln 2 - 1):1$ (E) NOTA

35. A cube is inscribed in a right circular cone of radius 2 and height 5 so that one of the faces is contained in the base of the cone. Find the surface area of the cube.

(A) $\frac{39600-24000\sqrt{2}}{289}$ (B) $\frac{22800-16000\sqrt{2}}{49}$
(C) $\frac{8100-3000\sqrt{2}}{529}$ (D) $\frac{25800-18000\sqrt{2}}{49}$ (E) NOTA

36. Evaluate: $\int_1^3 \frac{x+1}{x+2}\,dx$

(A) $\ln\left(\frac{625}{9}\right)$ (B) $2+\ln\left(\frac{3}{5}\right)$ (C) $\ln\left(\frac{25}{3}\right)-\frac{1}{2}$ (D) $\ln\left(\frac{25}{3}\right)$ (E) NOTA

37. Find the radius of convergence of the power series centered at $x = 0$:

$$1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\ldots+F_n x^n+\ldots$$

where F_n is the nth Fibonacci number.

(A) $\frac{2}{\sqrt{5}-1}$ (B) $\frac{\sqrt{5}}{2}+\frac{3}{34}$ (C) $\frac{-1+\sqrt{5}}{2}$ (D) $\frac{34}{3+17\sqrt{5}}$ (E) NOTA

38. Define the following values for R, the region bounded by the graphs of $y=\ln x$, $x = 1$, $y = 0$, and $x = e$:

A = volume of R revolved about the x-axis
B = volume of R revolved about the y-axis
C = area of R

Express the centroid of R in terms of these values.

(A) $\left(\frac{A}{C},\frac{B}{C}\right)$ (B) $\left(\frac{B}{2\pi C},\frac{A}{2\pi C}\right)$
(C) $\left(\frac{B}{C},\frac{A}{C}\right)$ (D) $\left(\frac{A}{2\pi C},\frac{B}{2\pi C}\right)$ (E) NOTA

39. Which of the following pairs is relatively prime for all natural numbers x?

(A) x and $x+2$ (B) $5x+1$ and $2x-4$
(C) $4x+5$ and $6x+7$ (D) $8x-5$ and $x+4$ (E) NOTA

40. In "Neo Cylindrical Coordinates," a point in space is determined by an ordered triple (r,θ,α), where r and θ are the polar coordinates of the point's projection onto the xy-plane and α is the angle between the position vector of the point and the position vector of the point's projection onto the xy-plane.. Express $\iiint \frac{dV}{x^2+y^2}$ in this coordinate system.

(A) $\iiint \frac{\sin^2\theta}{r^2}\,dr\,d\theta\,d\alpha$ (B) $\iiint \sec^2\alpha\ dr\,d\theta\,d\alpha$
(C) $\iiint r^2\sec^2\alpha\ dr\,d\theta\,d\alpha$ (D) $\iiint \sin^2\theta\ dr\,d\theta\,d\alpha$ (E) NOTA