Mu Alpha Theta National Convention: Denver 2001
Mu Individual Test – Solutions

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1. **(B)**. Collect all terms to one side to get $8x^2 - 10x - 3 = 0$. The sum of the roots is $-b/a = -(-10)/8 = 5/4$.

2. **(A)**. $\dfrac{3\sqrt{96}}{4} + 2\sqrt[3]{40} + \dfrac{\sqrt{216}}{3} + \dfrac{2\sqrt[3]{135}}{3} = 3\sqrt{6} + 4\sqrt[3]{5} + 2\sqrt{6} + 2\sqrt[3]{5} = 5\sqrt{6} + 6\sqrt[3]{5}$

3. **(B)**. Using $(6,3)$ as an origin, we create vectors with the other points to obtain $[-7,-1]$ and $[3, k-3]$. The three points are collinear if these two vectors are scalar multiples of each other, i.e. $3/(-7) = (k-3)/(-1)$. Solving, we get $k = 24/7 = m/n$ so $\sqrt{m^2+n^2} = \sqrt{24^2+7^2} = 25$.

4. **(A)**. $\dfrac{1}{8+15i} = \dfrac{1}{8+15i}\dfrac{(8-15i)}{(8-15i)} = \dfrac{8-15i}{8^2+15^2} = \dfrac{8-15i}{289}$

5. **(C)**. There are five good ways for the dice to come up $((2,6),(3,5),(4,4),(5,3),(6,2))$ out of the possible $6^2 = 36$ outcomes so the probability is $5/36$.

6. **(B)**. $x = 10^3 + 3 = 1003$ so x with its digits reversed is 3001.

7. **(C)**. Anything in the tens place, hundreds place, and so on, does not affect the units digit so we only consider the number 2^{111082}. Since the units digit of powers of two cycle with a period of 4 and $111082 \equiv 2 \pmod 4$, it follows that the units digit of 71182^{111082} is $2^2 = 4$. Hence, $U = 4$. Now because $221 = 13 \times 17$, $W = (1+1)(1+1) = 4$ and $UW = (4)(4) = 16$.

8. **(D)**. $\sin\arctan\dfrac{33}{56} = \sin\arcsin\dfrac{33/56}{\sqrt{(33/56)^2+1}} = \dfrac{33}{65}$

9. **(C)**. Cancelling all common factors from the numerator and denominator, we get $f(x) = (2x+7)/(5x+3)$. Thus, f has one vertical asymptote, $x = -3/5$, and one horizontal asymptote, $y = 2/5$, for a total of two.

10. **(B)**. The limit is just the derivative of $f(x) = x^4$ evaluated at $x = 2$, or $4(2)^3 = 32$.

11. **(D)**. Use your favorite technique for solving 3×3 systems of equations to obtain $(x,y,z) = (1,2,3)$. Thus, $x^2+y^2+z^2 = 1+4+9 = 14$.

12. **(B)**. Average Speed $= \dfrac{f(8)-f(2)}{8-2} = \dfrac{14-4}{6} = \dfrac{5}{3}$

13. **(C)**. The roots of the equation are $\{1, 2, i, -i\}$ so the sum of the real roots is 3. Note that this is equal to the sum of all the complex roots, as the imaginary solutions cancel each other out.

14. **(D)**. Since $y' = \ln x + x(1/x) - 1 = \ln x$ and $y'' = 1/x$, the values of the first and second derivative at $x = e^4$ are 4 and e^{-4}, respectively. Both values are positive so the graph is increasing and concave up and the given point.

15. **(D)**. The determinant of a product of matrices is the product of the determinants of each matrix factor, giving us $(16-24)(4-6)(7-15) = -128$.

16. **(A)**. Since $x^2+1 > x-1$ on the interval $0 < x < 1$, the area of R is given by $\int_0^1 (x^2+1) - (x-1)\,dx = 11/6$.

17. **(D)**. The locus is a line, namely, the one halfway between and parallel to the two lines.

18. **(C)**. We examine each claim:

    I. False. If $f(x) = 1/x$, $g(x) = -1/x$ and $c = 0$, the limits of each individual function as $x \to c$ do not exist but $\lim_{c\to 0}(f(x)+g(x)) = 0$.

    II. True. Differentiability implies continuity which requires that the function be defined at the particular value.

    III. False. Let $f(x) = x$. Then $\int_{-2}^{5} f(x)\,dx > 0$ but $f(x)$ is not always nonnegative on $-2 \le x \le 5$.

19. **(A)**. Drawing the diagram, we see that $\triangle DEF$ is comprised of $\triangle ABC$, an equilateral triangle congruent to $\triangle ABC$, and two $30° - 60° - 90°$ triangles with short legs of 2 and 1. Thus, the area of $\triangle DEF$ is $2(2^2\sqrt{3}/4) + (2)(2\sqrt{3})/2 + (1)(\sqrt{3})/2 = 9\sqrt{3}/2$.

20. **(D)**. Since $\cos x + \sin x > 0$ when $0 \le x \le \pi/4$, we can drop the absolute value sign and simplify the integrand to $(\cos 2x)/(\cos x + \sin x) = (\cos^2 x - \sin^2 x)/(\cos x + \sin x) = \cos x - \sin x$. The value of the integral is then $\int_0^{\pi/4} \cos x - \sin x\,dx = \sqrt{2} - 1$.

21. **(E)**. Generating the terms, we notice that $a_2 - a_1 = 4$, $a_3 - a_2 = -4/3$, and $a_4 - a_3 = 4/9$. Hence, the differences between two successive terms create a geometric sequence with first term 4 and common ratio of $-1/3$; its sum is $S = 4/(1-(-1/3)) = 3$. Consequently, $\lim_{n\to\infty} a_n = a_1 + S = 5 + 3 = 8$.

22. **(D)**. If $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, $P'(0) = 4a(0)^3 + 3b(0)^2 + 2c(0) + d = d$ and $P''(0) = 12a(0)^2 + 6b(0) + 2c = 2c$. The given information asserts that these two values are equal; thus, $d = 2c$ or $d/c = 2$, precisely what the problem is asking for.

23. **(C)**. From the many similar triangles formed, $CD = (BC)(AC)/AB = (6)(8)/(10) = 24/5$, $AD = (AC)^2/AB = (8)^2/10 = 32/5$, and $DE = AD - AE = 32/5 - 5 = 7/5$. The area of $\triangle CDE$ is $(1/2)(CD)(DE) = (1/2)(24/5)(7/5) = 84/25$.

24. **(C)**. For each equation, divide both sides by the coefficient of x (a legal move since the modulus is prime) and add 4 to both sides to obtain $x + 4 \equiv 7 \equiv 0 \pmod{7}$ and $x + 4 \equiv 13 \equiv 0 \pmod{13}$. These two equations combine to the single congruence $x + 4 \equiv 0 \pmod{91}$ which means $x \equiv -4 \equiv 87 \pmod{91}$.

25. **(A)**. We first find C. We have $f(x) = (x-1)+(x-2) = 2x-3$ for $x > 2$ so $f''(4) = 0$, making $C = 0$. When $1 < x < 2$, f will be at its minimum, or $(x-1)+(2-x) = 1$ which means $M = 1$. If $x < -4$, $f(x) = (1-x)+(2-x) = 3-2x$ so $f'(x) = N = -2$. The quantity we want is then $(M^2 - N^2 + IC)/2 = ((1)^2 - (-2)^2 + I(0))/2 = -3/2$.

26. **(A)**. Note that $\sum_{n=4}^{8}\binom{n}{4} = \binom{9}{4} = \binom{9}{5} = \binom{126}{1} = \binom{126}{125} = \binom{x}{y}$. The smallest possible value of $x+y$ is $9+4 = 13$.

27. **(B)**. $\displaystyle\int_1^e x + \frac{2}{x}\,dx = \frac{x^2}{2} + 2\ln|x|\Big|_1^e = \frac{e^2+3}{2}$

28. **(D)**. By the Shell Method, the volume is given by

$$2\pi\int_{e^4}^{e^5} x(x\ln x)\,dx = 2\pi\int_{e^4}^{e^5} x^2\ln x\,dx = 2\pi\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_{e^4}^{e^5} = \frac{28\pi e^{15} - 22\pi e^{12}}{9}$$

29. **(D)**. The expression is equal to $(a^2+b^2)^4 = (2^2)^4 = 256$.

30. **(C)**. Let $a_n = n^2(2x-1)^n/2^n$. By the Ratio Test, the power series converges if $\lim_{n\to\infty}|a_{n+1}/a_n| < 1$. Substituting a_n, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2(2x-1)^{n+1}}{2^{n+1}}\frac{2^n}{n^2(2x-1)^n}\right| = \left|\frac{2x-1}{2}\right| < 1$$

Thus, $|2x-1| < 2$, or $-1/2 < x < 3/2 \to |b-a| = 2$.

31. **(B)**. Notice that $(3-5i)(2+7i) = 41+11i$ so the equation conveniently simplifies to $x(2+7i)+y(3-5i) = 5+2i$ or $(2x+3y)+(7x-5y)i = 5+2i$. Equate real parts and imaginary parts to obtain the system $2x+3y = 5$ and $7x-5y = 2$. Solving produces $(x,y) = (1,1)$; thus $x^2+2xy+y^2 = (x+y)^2 = 4$.

32. **(A)**. Let $u = 2x$ so that $du = 2\,dx$ and

$$I = \frac{1}{2}\int_{12}^{14} f(u)+g(u)\,du \geq \frac{1}{2}\left(\frac{3}{4}+\frac{7}{2}\right) = \frac{17}{8}$$

33. **(D)**. When $(x,y) = (4,2)$, $t = 2$. Since $dy/dx = (3t^2-3)/(2t)$, the slope of the tangent line at the given point is $(3(2)^2-3)/(2(2)) = 9/4$. Point-slope form yields $y - 2 = (9/4)(x-4) \to y = 9x/4 - 7$.

34. **(A)**. Observe that $a^n - 1 = (a-1)(1+a+a^2+\cdots+a^{n-1})$. If $a > 2$, then $a^n - 1$ would be divisible by an integer greater than 1. So for $a^n - 1$ to have any hope of being prime, a must equal 2.

35. **(C)**. Like problem 18, we tackle each statement separately:

    I. This is the definition of absolute convergence and is unarguably true.

    II. False. Let $a_n = b_n = (-1)^n/\sqrt{n}$. Individually, each infinite series converges but the series of their product, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} 1/n$, diverges.

    III. True. Rewrite the limit as $\lim_{n\to\infty} a_n/(1/2^n)$. This limit is equal to a positive number and since $\sum_{n=1}^{\infty} 1/2^n$ is a convergent series, $\sum_{n=1}^{\infty} a_n$ also converges by the Limit Comparison Test.

36. **(A)**. Recall that if g is the inverse of f, then $g'(x) = 1/f'(g(x))$. Differentiating this, we get $g''(x) = -g'(x)f''(g(x))/(f'(g(x)))^2 = -f''(g(x))/(f'(g(x)))^3$. Since $f(1) = 2$, $g(2) = 1$, and $|g''(2)| = |-f''(1)/(f'(1))^3| = |-6(1)/(3(1)^2+2)^3| = 6/125$.

37. **(C)**. Because set operations are distributive, $(C \cup A^c) \cap (C \cup B^c) = C \cup (A^c \cap B^c)$. By DeMorgan's Theorem, $C \cup (A^c \cap B^c) = C \cup (A \cup B)^c$.

38. **(A)**. Apply the Chain Rule, to obtain $h'(x) = g'(x)f'(g(x))$. Using the given functions to gather the necessary values, we get $h'(0) = g'(0)f'(3) = (-2)(16) = -32$.

39. **(D)**. Keep using L'Hôpital's Rule until the indeterminate forms stop:

$$\lim_{x\to 2} \frac{4x^2 - 16x + 16}{3x^3 - 9x^2 + 12} = \lim_{x\to 2} \frac{8x - 16}{9x^2 - 18x} = \lim_{x\to 2} \frac{8}{18x - 18} = \frac{4}{9} = \frac{m}{n}$$

Thus, $m + n = 4 + 9 = 13$.

40. **(B)**. From the double-angle formula $2\cos^2 x - 1 = \cos 2x$, we find that $f(n) = 2n - 1$. Thus, $\sum_{n=1}^{16} f(n/32) = \sum_{n=1}^{16}(n/16 - 1) = -15/2$.

41. **(C)**. Let $u = 2-3x^2$. Then $du = -6x\,dx$ and the integral becomes $\int_0^{\sqrt{6}/3} x\sqrt{2-3x^2}\,dx = \int_2^0 -\sqrt{u}/6\,du = \int_0^2 \sqrt{u}/6\,du = 2\sqrt{2}/9$.

42. **(B)**. Multiply both sides of the equation by x and rearrange to get $x^2 - x + 1 = 0$. Using the quadratic formula, the solutions are $x = (1 \pm i\sqrt{3})/2$. For now, let $x = (1 + i\sqrt{3})/2$ or after converting to exponential form, $x = e^{i\pi/3}$ (as you'll see later, this will not make a difference). By De Moivre's Theorem, $x^n = e^{ni\pi/3}$ and $x^{-n} = e^{-ni\pi/3}$, making $x^n + x^{-n} = x^n + 1/x^n = e^{ni\pi/3} + e^{-ni\pi/3} = 2\cos\frac{n\pi}{3}$. We can now write the given equation in terms of the cosine rather than x:

$$\cos\frac{a\pi}{3} + \cos\frac{b\pi}{3} + \cos\frac{c\pi}{3} = 0$$

Playing around with small values leads to the minimal solution $(a, b, c) = (1, 3, 5)$ (or any permutation of these, for that matter) so $a + b + c = 9$.

43. **(B)**. Clear out all fractions so that $(2y+3x)(4x+7y)-3y-2x = 12x^2+29xy+14y^2-2x-3y = 0 = F(x,y)$. Using the method of partial derivatives, $dy/dx = -F_x(x,y)/F_y(x,y) = -(24x+29y-2)/(29x+28y-3) = (24x+29y-2)/(3-28y-29x)$.

44. **(B)**. Let the common ratio of S equal 2^k, where k is a positive integer. By the nth term formula for a geometric sequence, $a_n = ar^{n-1} = (1)(2^k)^{n-1} = 1024 = 2^{10}$. Equate powers of 2 and obtain the equation $k(n-1) = 10$. The only solution such that n is even and greater than 2 is $(n,k) = (6,2)$. Thus, the sum of the terms in S is $a(r^n-1)/(r-1) = (1)((2^2)^6-1)/(2^2-1) = 1365$.

45. **(D)**. Apply trigonometric substitution with $\sin u = x$, $\cos u\, du = dx$, and $\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$ (we assume that $-\pi/2 \le u \le \pi/2$ for this to work). Now change everything in terms of u, evaluate the (simpler) integral, and convert back to x:

$$\int \sqrt{1-x^2}\,dx = \int \cos u\,(\cos u\,du) = \int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{2}\sin u\cos u + C$$
$$= \frac{1}{2}\arcsin u + \frac{1}{2}x\sqrt{1-x^2} + C$$

46. **(E)**. Let $\#(w)$ be the number of ways to arrange the letters such that the word w appears. Treating a word as a single unit, we get $\#(\text{MATH}) = 6!$, $\#(\text{IS}) = 8!$, and $\#(\text{FUN}) = 7!$. Similarly, $\#(\text{MATH} \cap \text{IS}) = 5!$, $\#(\text{MATH} \cap \text{FUN}) = 4!$, and $\#(\text{FUN} \cap \text{IS}) = 6!$. The number of ways all three words appear in the arrangement is $\#(\text{MATH} \cap \text{IS} \cap \text{FUN}) = 3!$. By the Principle of Inclusion-Exclusion, the number of arrangements so at least one of the words appear is $6!+8!+7!-5!-4!-6!+3! = 45222$.

47. **(A)**. Using the Angle Bisector Theorem, $CB = (AC)(DB)/AD = (5)(6)/3 = 10$. By Stewart's Theorem, $(AC)^2(DB)+(CB)^2(AD) = (CD)^2(AB)+(AD)(AB)(DB)$. Plug in all known values into the formula and solve to get $CD = \sqrt{32} = 4\sqrt{2}$.

48. **(C)**. A square matrix will not have an inverse when its determinant is zero. In other words, we need to solve the equation $(x-2)(x-3)-2x(x+5) = -x^2-15x+6 = 0$. The discriminant of this quadratic is $b^2-4ac = (-15)^2-4(-1)(6) > 0$ so the equation has 2 distinct real solutions.

49. **(A)**. Let r be the radius of the circle. By Power of a Point, $(CE)(EM) = (AE)(EB)$ or $(4)(3) = 12 = (r+\sqrt{16-r^2})(r-\sqrt{16-r^2}) = r^2-(16-r^2) = 2r^2-16$. Thus, $r^2 = 14$ so the area of the circle is 14π.

50. **(A)**. Given a regular tetrahedron with side length s, the volume V is equal to $V = s^3\sqrt{2}/12$. Differentiating with respect to time t and using the fact that $s = 2$ when one face area is $\sqrt{3}$, we get

$$\frac{dV}{dt} = 6 = \frac{s^2\sqrt{2}}{4}\frac{ds}{dt} = \sqrt{2}\frac{ds}{dt} \rightarrow \frac{ds}{dt} = 3\sqrt{2}$$