Mu Alpha Theta National Convention: Denver 2001
Theta Individual Test – Solutions

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1. **(B)**. Since $\triangle ACB$ is isosceles, $\angle CAB = 180° - 2(50°) = 80°$. Thus, $\angle DAB = 60° + 80° = 140°$ and because $\triangle DAB$ is isosceles, $\angle ADB = (180° - 140°)/2 = 20°$.

2. **(A)**. The ratio of seawater to everything is $2 : 5$ so we have $(2/5)(20) = 8$ ml of seawater.

3. **(E)**. $\sqrt{x^{3/4}}(x^6)^{1/3} = x^{3/8}x^2 = x^{19/8}$

4. **(A)**. $9 \le x \le 12 \to 3 \le x/3 \le 4 \to 2 \le x/3 - 1 \le 3 \to b - a = 1$

5. **(C)**. $i - 2i^2 + 3i^3 - 4i^4 = i + 2 - 3i - 4 = -2 - 2i = a + bi \to a^2 + b^2 = 8$

6. **(B)**. To create a 0 at the end of a number, we have to multiply by $10 = 5 \times 2$. In the given number, there are fewer 2s than 5s so the number of zeroes is given by the exponent of 2, in this case, 19.

7. **(C)**. Let c and b equal the cost of a page of color and black-and-white prints, respectively. Then from $c = 3b$ and $c + 2b = 50$, we can deduce that $b = 10$ and $c = 30$. Printing an additional 50 charts will cost $(50)(30) = 1500$ cents, or $15.00.

8. **(D)**. We have an arithmetic sequence with a first term of -2 and common difference 3. Thus, the 49th term is $-2 + (48)(3) = 142$.

9. **(C)**. By definition, $\log_x y = z$ if and only if $x^z = y$.

10. **(B)**. The standard equation for a circle centered at (h, k) with radius of r is $(x-h)^2 + (y-k)^2 = r^2$. Let $(h, k) = (-3, 1)$ and $r = \sqrt{5}$ to get $(x+3)^2 + (y-1)^2 = 5$.

11. **(E)**. 147 is divisible by 3, 145 is obviously composite, 143 is divisible by 11, and 141 is divisible by 3, so none of these answers work (the answer is 139).

12. **(B)**. Using $30° - 60° - 90°$ triangles, we find that the ratio of the radius of C_1 to C_2 is 2. Thus, their areas are in a ratio of $2^2 = 4$.

13. **(C)**. Since Clymer drove the car for 14 days and drove an excess of 600 miles, the cost of the lease—including the down payment—is $50 + 14(30) + 600(.10) = 530$ dollars.

14. **(D)**. Use synthetic division; the answer is the remainder:

$$\begin{array}{r|rrrrrr} 2 & -3 & 3 & -9 & 1 & 2 & 5 \\ & & -6 & -6 & -30 & -58 & -112 \\ \hline & -3 & -3 & -15 & -29 & -56 & -107 \end{array}$$

15. **(D)**. Let the integers be n, $n+1$, and $n+2$. The problem states that $n/3+(n+1)/2+2(n+2)=n+(n+1)+(n+2)$. Solve to get $n=9$; the numbers are 9, 10, and 11.

16. **(A)**. The units digit of p^k, where p is prime, cycle with a period of 4. Furthermore, we can ignore 11^{268} since its units digit is 1. Replacing the powers of each prime with its remainder when divided by 4, we get a number with the same units digit as the original: $2^3 3^1 7^1 = 168$. Its units digit is 8.

17. **(D)**. $\log_7 \frac{1}{343} + \log_4 1024 - \log_5 \frac{1}{25} = -3+5+2=4$

18. **(D)**. There are 20 ways to pick a woman and 19 ways to pick a man that's not her husband. The total number of ways is $(20)(19)=380$.

19. **(A)**. For $f(x)$ to be defined, $x^2+x-2>0$ (it's important that finding the correct domain involves working with the *original* value of f; we just can't cancel out the e and natural log). Solving, we get $x<-2$ or $x>1$.

20. **(D)**. Since $81=9^2$ and $2401=49^2$, the squares inside the given interval are 10^2 and 48^2, giving us a total of $48-10+1=39$ squares.

21. **(B)**. $\begin{pmatrix} 4 & 2 \\ -8 & 9 \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 36+4 & -12+10 \\ -72+18 & 24+45 \end{pmatrix} = \begin{pmatrix} 40 & -2 \\ -54 & 69 \end{pmatrix}$

22. **(D)**. Place two quarters on both ends; now we're left with three nickels and two quarters. The number of ways to arrange the following is equal to $(5!)/(3! \times 2!) = 10$.

23. **(C)**. There are five ways for an even and odd to sum to 11: 1+10, 3+8, 5+6, 7+4, and 9+2. Without restriction, there are $(5)(5)=25$ ways of choosing an even and odd integer. The probability is $5/25=1/5$.

24. **(A)**. We have $1/x+1/y=(x+y)/xy=7/(xy)=3/8$ so $xy=56/3$. Now obtain the needed quantity by writing $x^2+xy+y^2=(x+y)^2-xy=(7)^2-(56/3)=91/3$.

25. **(C)**. Collect everything to the greater side of the equation and simplify to obtain $(14x-11)/((x+1)(x-4))>0$. Test around the critical values of -1, $11/14$, and 4 to find that the solution set is $-1<x<11/14$ or $x>4$.

26. **(A)**. The sum of the binomial coefficients is 2^{1024}, hence $\log_4 2^{1024} = \log_4 4^{512} = 512$.

27. **(A)**. From the given constraint, either two cases can happen: picking a fair coin and having it land heads, or picking a double-headed coin and having it land heads for a total probability of $(6/10)(1/2)+(4/10)(1)=7/10$. The probability of choosing a double-headed coin and having it land heads is $(4/10)(1)=4/10$. Thus, the value of the conditional probability is $(4/10)/(7/10)=4/7$.

28. **(C)**. To get from $(x-y)(x+y)=y(x-y)$ to $x+y=y$, we must divide both sides of the equation by $x-y$, not allowed because $x-y=0$.

29. **(A)**. In standard form, the equation of the conic is $(x-1)^2/1 - (y-1)^2/4 = 1$, a hyperbola. When it's rotated 90 degrees clockwise about the origin, the hyperbola now points up-down, the center gets reflected about the x-axis, and the axes lengths switch, i.e. the numbers in the denominators of the squared terms. The new equation is $(y+1)^2 - (x-1)^2/4 = 1$.

30. **(D)**. Simplify each side of the equation to obtain the identity $(4x+3)/(2x+1) = (4x+3)/(2x+1)$. Thus, the probability is 1.

31. **(C)**. Arrange the test scores in increasing order, making the fourth element be 40. Since 28 is the mode, it must come up at least twice; let it be the first and second test score. Since our goal to minimize the seventh test score, we want to maximize the values of every other element before that. The third score is at most 39 because if it were 40 then there'd be two modes, a contradiction. The sum of the last three scores must then be equal to $(51)(7) - 2(28) - 39 - 40 = 222$. Play around with some numbers to find that the smallest possible value for the seventh score is 75.

32. **(B)**. Consider the function $f(x) = 4$. The inverse of f does not exist (it fails the horizontal line test) and the range of f is 4. Its domain, however, is the set of all real numbers, true of all linear functions.

33. **(B)**. $2^{10} = 1024 > 1000$ which means $2^{20} > 1000^2 = 1000000$ so $k = 19$.

34. **(A)**. The four pieces cut from each corner are all in the shape of isosceles right triangles, the hypotenuse of which equals the side length of the octagon. Call this length x. Then the side length of the square that remains after truncation is $5 - 2(x/\sqrt{2}) = 5 - x\sqrt{2}$. This is also equal to the side length of the octagon, giving us $5 - x\sqrt{2} = x$. Solve to get $x = 5\sqrt{2} - 5$.

35. **(C)**. Factor the first equation to get $(8a+9b)(8a-9b) = (126)(8a-9b) = 2268$ so $8a - 9b = 18$. Using this with the third equation, we get $(a,b) = (9,6)$. Check this against the second equation just to be sure: $9^2 - 6^2 = 81 - 36 = 45$. Indeed, it works! Thus, $a^2 + b^3 = 9^2 + 6^3 = 297$.

36. **(D)**. The solid of maximal volume will be a cube with edge length $\sqrt{24/6} = 2$. Therefore, the volume is 8.

37. **(D)**. There are $\binom{4}{1}$ ways to pick an ace, $\binom{4}{1}$ ways to pick a king, and $\binom{52}{2}$ ways to pick without restriction. The probability is $\binom{4}{1}\binom{4}{1}/\binom{52}{2} = 8/663$.

38. **(C)**. The region in the problem is inside the triangle with vertices $(0,0)$, $(3,4)$, and $(9,0)$. We check lattice points in two cases. When $x \leq 3$, we look for how many integer values satisfy $0 < y < 4x/3$. If $x > 3$, we use the inequality $0 < y < -2x/3 + 6$. In each case, x is an integer from 1 to 9, inclusive. Using this technique, we find 13 lattice points.

39. **(A)**. Work in modulo 10 and partition the possible remainders upon division by 10 such that they add up to 0 (mod 10):

$$\{0\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$$

We will have a pair of integers whose sum or difference is a multiple of 10 if at least one of these six compartments is filled with two elements. By the Pigeonhole Principle, we need at least $6+1=7$ integers.

40. **(B)**. $a/(1-1/2)=27 \rightarrow a=27/2 \rightarrow a_3=ar^2=(27/2)(1/2)^2=27/8$

41. **(C)**. Notice that 360 equals $(6)(5)(4)(3)=(8-2)(8-3)(8-4)(8-5)$ and $(-3)(-4)(-5)(-6)=(-1-2)(-1-3)(-1-4)(-1-5)$. This implies that x equals -1 or 8—their sum is 7.

42. **(B)**. In $\triangle ABC$, let $AC=9$, $AB=7$, $BC=4$, and I the foot of the median to BC. From A, go outward to I and create line segment AD such that $ABDC$ is a parallelogram. From the fact that the diagonals of a parallelogram bisect each other, $AI=ID=$ length of median. Moreover, since the sum of the squares of the sides equals the sum of the squares of the diagonals, we have $9^2+7^2+9^2+7^2=4^2+(2AI)^2$. Solving produces $AI=\sqrt{61}$.

43. **(B)**. Compute the magnitude of numerator and denominator separately and divide: $|10-24i|/|4+3i|=26/5$.

44. **(B)**. Some painful arithmetic yields $a_2=.48$, $a_3=.4992$, and $a_4=.49999872$. Thus, $.5-a_4=.00000128$, or 1.28×10^{-6}.

45. **(D)**. 224_7 in base ten is $(2)7^2+(2)7+(4)1=116$. Suppose $116=a_4 3^4+a_3 3^3+a_2 3^2+a_1 3+a_0$. Taking both sides modulo 3, we get $2\equiv a_0 \pmod 3$ hence, $a_0=2$. Eliminate a_0 from the first equation and proceed similarly (except take both sides modulo 3^2) to obtain the representation $116=(1)3^4+(1)3^3+(0)3^2+(2)3+(2)$. Thus, 116 in base three is 11022.

46. **(C)**. The function simplifies to $f(x)=1+1/x$. Since f is strictly decreasing for all real numbers x, f will attain its minimum at the right endpoint of the interval, or $x=2$. Thus, the minimum value is $f(2)=3/2$.

47. **(B)**. By the Angle Bisector Theorem, $AB/AD=BC/DC \rightarrow 4/AD=6/(5-AD)$. Solve and get $AD=2$.

48. **(C)**. List them out (or find the coefficient of x^{14} of the generating function

$$\left(\sum_{p\in\{2,3,5,7\}} x^p\right)\left(\sum_{i=1}^{5} x^{2i-1}\right)\left(\sum_{i=0}^{4} x^{2i}\right)$$

the former being the more efficient route). There are 13 that have a digital sum of 14.

49. **(B)**. The region formed by $|x| + |y| < 20$ is a square with side length $20\sqrt{2}$ so its area is $(20\sqrt{2})^2 = 800$.

50. **(A)**. Given a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is defined by $|A| = ad - bc$. Furthermore, the determinant of a product of matrices is the product of the determinants of each individual matrix factor. Thus, $|D| = |A||B||C| = (-5)(10)(2) = -100$.