Mu Alpha Theta National Convention: Denver, 2001
Matrices and Vectors Topic Test Solutions – Alpha Division

1. $\begin{bmatrix} 1-2 & -2-^-2 & 3-^-1 \\ -4-^-4 & 1-^-3 & -3-6 \\ 1-1 & -2-^-2 & 0-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 4 & -9 \\ 0 & 0 & -2 \end{bmatrix}$ B

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2. $2\cdot 1-4\cdot y=7$ $\quad 2\cdot{}^-3-4\cdot{}^-2=2$ ✓
$2\cdot x-4\cdot{}^-1=4$ $\quad 2\cdot 2-4\cdot 3=-8$ ✓

$-4y=5$ $\quad 2x=0$
$y=\frac{-5}{4}$ $\quad x=0$ $\quad x+y=\frac{-5}{4}$ D

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3. $\begin{bmatrix} 5\cdot{}^-1+0\cdot 3 & 5\cdot{}^-5+0\cdot 0 & 5\cdot 4+0\cdot{}^-2 \\ 3\cdot{}^-1+2\cdot 3 & 3\cdot{}^-5+2\cdot 0 & 3\cdot 4+2\cdot{}^-2 \\ 1\cdot{}^-1+{}^-4\cdot 3 & 1\cdot{}^-5+{}^-4\cdot 0 & 1\cdot 4+{}^-4\cdot{}^-2 \end{bmatrix} = \begin{bmatrix} -5 & -25 & 20 \\ 3 & -15 & 8 \\ -13 & -5 & 12 \end{bmatrix}$ B

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4. $\underset{A}{\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}} \underset{B}{\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{bmatrix}} = \underset{AB}{\begin{bmatrix} & & 0+9+24 \\ & & \\ 8+25+48 & & \end{bmatrix}}$ $\quad 33+81=114$ D

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5. $-5\cdot\begin{vmatrix} -1 & -2 \\ 4 & 0 \end{vmatrix} + 5\cdot\begin{vmatrix} 3 & -1 \\ a & 4 \end{vmatrix} = -5\cdot 8+5(12+a)$
$= -40+60+5a = 5a+20$ A

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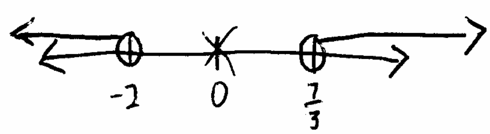
6. $-12+x(3x-1)>2$ $\quad\to (3x-7)(x+2)>0$
$3x^2-x-14>0$



C

7. switching two adjacent rows or columns changes the sign.
multiplying a row or column changes the sign.
consider A to be our baseline:
B involves 3 row flips and a -1, so its equal.
C involves a diagonal flip, the "transpose", which intuitively doesn't change anything, so it's equal.
D. involves one column flip, so it's unequal D

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8. To have exactly one solution, a system's matrix must have a non-zero determinant.

$$\begin{vmatrix} -2 & 1 & -5 \\ 1 & -2 & 1 \\ 3 & -A & 0 \end{vmatrix} = 3\cdot\begin{vmatrix} 1 & -5 \\ -2 & 1 \end{vmatrix} + A\begin{vmatrix} -2 & -5 \\ 1 & 1 \end{vmatrix}$$

$$= -27 + 3A = 0 \Rightarrow A = 9 \qquad B$$

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9. $r+s-t=-2$
$2r-2s+0t=5 \Rightarrow r-s=\frac{5}{2}$ B

10.
$$\frac{\begin{vmatrix} 3 & -1 & -3 \\ 2 & 4 & 1 \\ -5 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -2 \\ 2 & 4 & 0 \\ -5 & 0 & 5 \end{vmatrix}} = \frac{-5\begin{vmatrix} -1 & -3 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix}}{-2\begin{vmatrix} 2 & 4 \\ -5 & 0 \end{vmatrix} + 5\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix}} = \frac{-55+28}{-40+70} = \frac{-27}{30} = \frac{-9}{10} \qquad A$$

11. The cofactor is the signed determinant of what's left when an element's row & column are eliminated.

~~$-\begin{vmatrix} 4 & -5 \\ 2 & -3 \end{vmatrix} = 2$ C~~ OOps... that's first column, second row...

$-\begin{vmatrix} -2 & -4 \\ -1 & -3 \end{vmatrix} = -2$ B

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12. The inverse of a matrix is its adjoint divided by its determinant. The adjoint is the transposed matrix of cofactors.

$$\frac{1}{-4}\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$ B

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13. $c_{i,j} = a_{1,i} b_{1,j} + a_{2,i} b_{2,j}$

this looks like how we do matrix products, except instead of going across a row & down a column, we're going down two colums, so

$C = A^T B.$ B

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14. $\frac{1}{3-3z}\begin{bmatrix} -3 & -3 \\ 1 & z \end{bmatrix}$ sum of elements $= \frac{z-5}{3-3z}$ D

15. An eigenvector for a matrix is a vector which points the same direction before & after being transformed. An eigenvalue is the factor by which the ~~size~~ magnitude of an eigenvector is multiplied during the transformation. So, for matrix A, & eigenvector v, & eigenvalue λ, $Av = \lambda I v$, where I is the identity matrix. So, $(A-\lambda I)v = 0$, & $|A-\lambda I| = 0$.

$$\begin{vmatrix} 4-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0 = 12 - 7\lambda + \lambda^2 + 2$$

$$\lambda^2 - 7\lambda + 14 = 0 \quad \text{product} = 14 \qquad A$$

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16. $$\begin{bmatrix} -4-\lambda & 1 \\ 3 & -2-\lambda \end{bmatrix} = 8 + 6\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$(\lambda+5)(\lambda+1) = 0$$

$$\lambda = -5, -1$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \text{none of I, II, III, or IV}$$

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \text{III} \qquad B$$

17. One method of finding the inverse of a matrix is to ~~apply the~~ write them (your matrix & the identity matrix) out side by side and apply the same series of row operations to each of them until your matrix has been transformed into the identity matrix. At this point, the identity matrix will have been transformed into the inverse of the matrix of interest. So, if you only reduce M to $2I$, then I was only transformed to $2M^{-1}$! If you started with $12I$, then you'd end at $24M^{-1}$. C

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18. Multiplying a row or column of a matrix by a constant multiplies the determinant by that number. We're multiplying four rows by 4, so we multiply the determinant by 4^4, getting $4^5 = 2^{10} = 1024$. D

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19. M^{-1} would transform $\begin{bmatrix} y \\ z \\ x \\ w \\ v \end{bmatrix}$ into $\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$, so must be

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

so it will transform $\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$ into $\begin{bmatrix} z \\ y \\ x \\ v \\ w \end{bmatrix}$. D

20. The rank of a matrix is the order of the largest square sub-matrix with a non-zero determinant. Because the third row of the matrix is the sum of the first two, the answer must be less than three. Because the second row is not a multiple of the first, the answer is 2. D

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21. We're looking for all points P such that vector $\overrightarrow{AP}$ is a multiple of vector $\overrightarrow{AB}$. So, $\overrightarrow{AP}$ points the same direction (or opposite) as $\overrightarrow{AB}$, and thus P lies on the line connecting A & B. D

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22. Velocity is a vector, not a scalar: it contains direction as well as magnitude. Speed measures only magnitude, and is a scalar. Because the path taken is not specified, the average speed cannot be determined (the particle could have gone around the Earth several times before ending just 5 units away). We know that it moved 4 & 3 units in five seconds, however, so its average velocity is $\left[\frac{4}{5}, \frac{3}{5}\right]$. D

23. $$\begin{array}{r} 2[2,-3,1] \\ +\ [1,2,-1] \\ -4[-3,-1,2] \\ \hline [17,0,-7] \end{array}$$ D

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24. $\sqrt{-1^2+3^2+4^2} = \sqrt{26}$ C

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25. $1 \cdot -3 + 4 \cdot -4 = -19$ A

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26. $A \cdot B = |A||B|\cos\theta = \sqrt{3^2+-5^2}\,\sqrt{-2^2+1^2}\,\cos\theta = 3 \cdot -2 + -5 \cdot 1$

$\cos\theta = \frac{-11}{\sqrt{34}\sqrt{5}} \Rightarrow \theta \approx 147.5°$ D

alas, needs a calculator.

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27. If A & B are endpoints of a diameter of a circle, then P can be any point on the circle. Or, you can draw a right triangle APB and realize that the length of the median to AB is one-half the hypotenuse. B

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29. $\begin{vmatrix} i & j & k \\ 2 & 0 & -4 \\ -2 & 3 & 2 \end{vmatrix} = 12i + 4j + 6k$ D

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28 $\vec{a} = m[1,1,1]$ $\vec{i} = [1,0,0]$ $\cos\theta = \frac{1}{\sqrt{3}\sqrt{1}} \Rightarrow \theta \approx 54.7$ A

calculator again...

30. $\frac{2}{a} = \frac{-6}{-3} = \frac{-8}{4a} = r$

$r = 2 \Rightarrow a = 1,\ a = -1$

can't be done! E

31. The vector between the points is $[2,4,6]$, and $(1,1,2)$ is on the line, so A works.

32. $s[1,2] + t[-3,1] = [3,12]$

$$\begin{aligned} s - 3t &= 3 \\ 2s + t &= 12 \\ 2s - 6t &= 6 \\ \hline 7t &= 6 \end{aligned}$$

$t = \frac{6}{7} \Rightarrow s = \frac{39}{7}$

$s + t = \frac{45}{7}$ B

33. A plane has x, y & z coefficients equal to the components of its perpendicular vector.

$x - 2y + 3z = d = 2 + 6 - 12 = -4$

A

34. The vectors $[0,0,-5]$ & $[9,7,-2]$ lie in the plane.

So, their cross product, $[35,-45,0]$ is perpendicular to it.

$7x \cancel{+} + 9y = \cancel{4?}\,29$ D

35. The vector $[2,1,-5]$ is perpendicular to the plane. So, the point in the plane closest to $(-1,3,2)$ lies on the line $[x,y,z] = [-1,3,2] + t[2,1,-5]$. To get t, we solve $2(-1+2t) + (3+t) - 5(2-5t) = 12$

$30t - 9 = 12 \Rightarrow t = \frac{7}{10} \Rightarrow d = \frac{7}{10}\sqrt{2^2+1^2+{-5}^2} = \frac{7\sqrt{30}}{10}$ A

36. The line of intersection lies in both planes, and thus is perpendicular to both perpendicular vectors, and points in the $\begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ -3 & 1 & -1 \end{vmatrix} = [3,4,-5]$ direction. It will cross ~~some~~ the yz plane where $x=0$, meaning

$$\begin{aligned} -2y-z&=4 \\ y-z&=7 \\ \hline 3y&=3 \\ y=1 &\Rightarrow z=-6 \end{aligned}$$

B

37. To be parallel to the plane, their direction vectors must be perpendicular to the vector perpendicular to the plane, so their dot product must be zero.

$[6,2,-3] \cdot [3,1,7]$: $18+2+-21=-1$ ✗

$[6,2,-3] \cdot [1,-3,0]$: $6+-6+0=0$ ✓

$[6,2,-3] \cdot [2,3,6]$: $12+6+-18=0$ ✓

D

38. The equidistant points are in a plane perpendicular to the segment connecting them, & through its midpoint.

$[-1,-7,-6]$, $\left(\frac{5}{2}, \frac{1}{2}, -2\right) \Rightarrow x+7y+6z=-6$ B

39. $A=\frac{1}{2}ab\sin\theta=\frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{AC}\right|=\frac{1}{2}\left\|\begin{matrix} i & j & k \\ 6, & -6 & -4 \\ 7 & -10 & -1 \end{matrix}\right\|=\frac{1}{2}\left|-34i-22j-18k\right|$

$=\frac{1}{2}\sqrt{34^2+22^2+18^2}=\sqrt{491}$ C

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40. 6 seconds $\Rightarrow$ 12 units $= t\sqrt{4+4+1} = 3t \Rightarrow t = 4$

$[4,1,7] + [8,-8,-4] = [12,-7,3] \Rightarrow (12,-7,3)$ A