Mu Alpha Theta National Convention: Denver 2001
Advanced Calculus Topic Test – Solutions

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1. **(B)**. $\displaystyle\int_0^{\pi/2} \cos^2 t\, dt = \frac{1}{2}t + \frac{1}{4}\sin 2t \Big|_0^{\pi/2} = \frac{\pi}{4}$

2. **(A)**. By direct substitution, $\displaystyle\lim_{(r,\theta)\to(-1,3)} \frac{r^2 + 2\theta - r}{\theta^2 - 7r^5} = \frac{(-1)^2 + 2(3) - (-1)}{(3)^2 - 7(-1)^5} = \frac{1}{2}$.

3. **(B)**. Treating u as a constant and differentiating, $\partial M/\partial v = 2u^2 + 3u \sin uv$.

4. **(D)**. Rewrite the equation as $x^2 + \sin y - xy^3 = 0 = F(x, y)$. Since $dy/dx = -F_x/F_y$, we have $dy/dx = -(2x - y^3)/(\cos y - 3xy^2) = (2x - y^3)/(3xy^2 - \cos y)$.

5. **(C)**. $A_a = b/2$ and $A_b = a/2$ so $A_a A_b = ab/4 = A/2$.

6. **(C)**. We have n variables so there are clearly $n!$ orders of integration.

7. **(D)**. Differentiating with respect to y, we get $z_y = 3y^2 - 2x$. Thus, the answer is $3(2)^2 - 2(1) = 10$.

8. **(A)**. Recall that the mixed partial derivatives f_{xy} and f_{yx} are equal, making their ratio equal to 1.

9. **(C)**. If m_x and m_y are the moments about the x and y axes respectively and the mass of the region is M, then the center of mass is given by $(\overline{x}, \overline{y}) = (m_x/M, m_y/M)$. These values are calculated as follows: $m_x = \iint_R y\rho(x, y)\, dA = \int_0^1 \int_0^{1-x} y(x + y)\, dy\, dx = 1/2$, $m_y = \iint_R x\rho(x, y)\, dA = \int_0^1 \int_0^{1-x} x(x + y)\, dy\, dx = 1/2$, and $M = \iint_R \rho(x, y)\, dA = \int_0^1 \int_0^{1-x} x + y\, dy\, dx = 1/3$. Thus, $(\overline{x}, \overline{y}) = (3/8, 3/8)$.

10. **(B)**. We see that $3x \le y \le 4 - x^2$ and $0 \le x \le 1$ so using the order $dy\, dx$, the double integral is $\iint_R dA = \int_0^1 \int_{3x}^{4-x^2} dy\, dx$.

11. **(A)**. By the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (3x^2)(2\cos t) + (4y)(-10 \sin 2t) = 6x^2 \cos t - 40y \sin 2t$$

Substituting the proper values, we get $6(1)^2(\sqrt{3}/2) - 40(5/2)(\sqrt{3}/2) = -47\sqrt{3}$.

12. **(C)**. We have $D_x(x, y) = 2xy$ and $D_y(x, y) = x^2$ so $D_x(2, 0) + D_y(0, 1) = 0 + 0 = 0$.

13. **(D)**. Setting z-values equal to each other, we find that the solids intersect in a circle with equation $x^2 + y^2 = 9$. Since $18 - x^2 - y^2 \geq x^2 + y^2$ in this region, the volume of the solid is given by the double integral

$$\iint_R (18 - x^2 - y^2) - (x^2 + y^2)\, dA = \iint_R (18 - 2x^2 - 2y^2)\, dA$$

where R is the circle centered at the origin with radius 3. Converting to polar coordinates, we see that $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. Therefore $\iint_R (18 - 2x^2 - 2y^2)\, dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2)\, r\, dr\, d\theta = 81\pi$.

14. **(B)**. $\nabla T(x, y) = T_x\mathbf{i} + T_y\mathbf{j} = (42x^6 - 3y)\mathbf{i} + (-2\cos y \sin y - 3x)\mathbf{j} = (42x^6 - 3y)\mathbf{i} - (2\sin y + 3x)\mathbf{j}$.

15. **(A)**. Rewrite the equation as $2x^3 - y^2 + 9\sqrt{z} - 25 = 0 = F(x, y, z)$. Computing the partial derivatives in all directions, we get $F_x(x, y, z) = 6x^2 \to F_x(1, 2, 9) = 6$, $F_y(x, y, z) = -2y \to F_y(1, 2, 9) = -4$, and $F_z(x, y, z) = 9/(2\sqrt{z}) \to F_z(1, 2, 9) = 3/2$. The equation of the tangent plane is then given by $6(x-1) - 4(y-2) + (3/2)(z-9) = 0$ or $12x - 8y + 3z = 23$.

16. **(C)**. Let $f(x, y, z) = x^2 + y^2 + z^3 - 6$ and $g(x, y, z) = x - y - z$; thus, $\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 3z^2\mathbf{k}$ and $\nabla g(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$. The vector $\nabla f(2, 1, 1) \times \nabla g(2, 1, 1) = (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} - \mathbf{j} - \mathbf{k})$ gives us the direction vector for the line. Evaluating the cross product, we get $\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$. Thus, a set of parametric equations for the line is $x = 2 + u$, $y = 1 + 7u$, and $z = 1 - 6u$. Let $u = -t$ to obtain the answer in choice C.

17. **(C)**. Since $z = 8 - 2x + y$, the surface area is given by $\iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA = \iint_R \sqrt{1 + (-2)^2 + (1)^2}\, dA = \sqrt{6}(\text{Area of } R)$. Note that $|x| + |y| \leq 6$ is a square with side length $6\sqrt{2}$. Thus, the surface area is $\sqrt{6}(6\sqrt{2})^2 = 72\sqrt{6}$.

18. **(C)**. The directional derivative of $z(x, y)$ in the direction of a unit vector $\mathbf{u}$ at (x_0, y_0) is given by $\nabla z(x_0, y_0) \cdot \mathbf{u}$. Thus, we have $\nabla z(x, y) = \mathbf{i}/(1 + x^2) + \mathbf{j}/(1 + y^2)$ and $\mathbf{u} = (24/25)\mathbf{i} + (7/25)\mathbf{j}$. Letting $(x_0, y_0) = (1, 1)$, we get $(\mathbf{i}/2 + \mathbf{j}/2) \cdot ((24/25)\mathbf{i} + (7/25)\mathbf{j}) = 31/50$.

19. **(D)**. If $(s, t) = (0, 1)$, then $(x, y, z) = (0, 1, 1)$. By the Chain Rule

$$\begin{aligned} \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} &= (y + z)(t) + (x + z)(te^{st}) + (y + x)(0) \\ &= t(y + z) + t(x + z)e^{st} \end{aligned}$$

Substitute the given values to obtain $(1)(2) + (1)(1)(1) = 3$.

20. **(B)**. Drawing a graph, we find that $x^2 + 1 \leq y \leq 10$ and $-3 \leq x \leq 0$. It follows that the integral is equivalent to $\int_{-3}^{0} \int_{x^2+1}^{10} \sqrt{xy}\, dy\, dx$.

21. **(B)**. Let $m = r\cos\theta$ and $n = r\sin\theta$; then as $(m,n) \to (0,0)$, $r \to 0$. We can now write the limit as $\lim_{r\to 0}(r^3\cos^3\theta + r^3\sin^3\theta)/(r^2\sin^2\theta + r^2\cos^2\theta) = \lim_{r\to 0}(r\cos^3\theta + r\sin^3\theta) = 0$

22. **(C)**. Integrating each component of the vector field with respect to the proper variable, we get $\int 2xy+y^2\,dx = yx^2+xy^2+C(y)$ and $\int x^2+2xy\,dy = yx^2+xy^2+C(x)$. Comparing these two integrals, we see that $V(x,y) = yx^2 + xy^2$.

23. **(A)**. By the definition of divergence

$$\operatorname{div}\mathbf{G} = \frac{\partial}{\partial x}(zx^3) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(yz) = 3zx^2 + 0 + y = 3zx^2 + y$$

Thus, $\operatorname{div}\mathbf{G}(5,12,13) = 3(13)(5)^2 + 12 = 987$.

24. **(B)**. The first-order approximation in two variables is given by $h(a+\Delta a, b+\Delta b) \approx h(a,b) + h_a\Delta a + h_b\Delta b$. Because $h_a = a/\sqrt{a^2+b^2}$ and $h_a = b/\sqrt{a^2+b^2}$, we have $h(3+.1, 4+.1) \approx \sqrt{3^2+4^2} + (3/\sqrt{3^2+4^2})(.1) + (4/\sqrt{3^2+4^2})(.1) = 257/50$.

25. **(A)**. By the Arithmetic-Geometric Mean (AM-GM) Inequality

$$\frac{x^2/49 + y^2/16}{2} = \frac{1}{2} \geq \sqrt{\left(\frac{x^2}{49}\right)\left(\frac{y^2}{16}\right)} = \frac{xy}{28}$$

This implies that $xy \leq 14$, or $4xy \leq 56$.

26. **(D)**. Given a linear differential equation $y' + a(x)y = b(x)$, an integrating factor is given by $e^{\int a(x)\,dx}$. Here $a(x) = \ln x$ so $\int a(x)\,dx = x\ln x - x$ therefore, $e^{x\ln x - x} = x^x e^{-x}$.

27. **(D)**. Let $2x^2 + y^2 - 9x^4 - 8 = 0 = f(x,y,z)$. Since the gradient produces a vector that's normal to a surface, the value of $\nabla f(-2,3,1)$ is a direction vector for the desired line. We have $\nabla f(x,y,z) = 4x\mathbf{i} + 2y\mathbf{j} - 36z^3\mathbf{k}$ so $\nabla f(-2,3,1) = -8\mathbf{i} + 6\mathbf{j} - 36\mathbf{k}$ or, after dividing each component by -2, $4\mathbf{i} - 3\mathbf{j} + 18\mathbf{k}$. A set of symmetric equations is then $(x+2)/4 = (y-3)/(-3) = (z-1)/18$.

28. **(A)**. The projection of the solid onto the xy-plane is in the shape of a semicircle of radius 2 in the first and fourth quadrants. The height of the solid ranges from $0 \leq z \leq 1$ so the triple integral in cylindrical coordinates is $\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^1 r\,(r\,dz\,dr\,d\theta)$.

29. **(B)**. The infinite-dimensional integral is equivalent to $\prod_{n=2}^{\infty}(1 - 1/n^2)$. It's relatively easy to show by induction that $\prod_{n=2}^{k}(1 - 1/n^2) = (k+1)/(2k)$. Taking limits gives us $\lim_{k\to\infty}(k+1)/(2k) = 1/2$.

30. **(C)**. Notice that

$$\frac{\partial}{\partial y}(e^y + y^2 \cos x) = e^y + 2y\cos x = \frac{\partial}{\partial x}(xe^y + 2y \sin x)$$

which indicates that $\mathbf{W}(x, y)$ is a conservative vector field. Using the same technique in problem 22 to find potential functions, we get $W(x, y) = xe^y + y^2 \sin x$. By the Fundamental Theorem of Line Integrals, the answer is $W(3\pi/2, -1) - W(0, 0) = 3\pi/(2e) - 1$.

31. **(B)**. The Jacobian $\partial(x, y)/\partial(u, v)$ is given by the determinant of $\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$. Solving for x and y, we get $x = (u + v)/8$ and $y = (u - v)/2$. Thus

$$\frac{\partial(x, y)}{\partial(u, v)} = \left|\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{8}$$

32. **(D)**. Since $0 \le z \le 4 - y - 2x$, $0 \le x \le 2 - y/2 - z/2$. Project the solid into the yz-plane to obtain the bounds $0 \le y \le 4 - z$ and $0 \le z \le 4$. The triple integral is then $\int_0^4 \int_0^{4-z} \int_0^{2-y/2-z/2} \sqrt{xyz}\, dx\, dy\, dz$.

33. **(A)**. Let $M = 3x^2y + \cos x$ and $N = x^3 + 4xy^3 + \sin 5y$. Since C is simple closed curve, we can calculate the line integral using Green's Theorem rather than evaluating separately for each smooth path.

$$\oint_C M dx + N dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA = \iint_R (3x^2 + 4y^3) - (3x^2)\, dA = \iint_R 4y^3\, dA$$

where $R = \{(x, y) | \, x^2 \le y \le 1, 0 \le x \le 1\}$. Using the order $dy\, dx$, we have $\iint_R 4y^3\, dA = \int_0^1 \int_{x^2}^1 4y^3\, dy\, dx = \int_0^1 1 - x^8\, dx = 8/9$.

34. **(A)**. By Gauss' Divergence Theorem, the flux is given by

$$\iint_S \mathbf{S} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div} \mathbf{S}\, dV = \iiint_Q 2 + 1 - 2\, dV = \iiint_Q dV$$

which is just the volume of the solid formed by the surface; in this case, a sphere of radius 9. So the flux is equal to $4\pi(9)^3/3 = 972\pi$.

35. **(B)**. The roots of the characteristic equation $m^2 - 8m + 15 = 0$ are $m = 5$ and $m = 3$. Thus, the general solution is given by $y = ae^{3x} + be^{5x}$.

36. **(D)**. If $\mathbf{E}$ is a vector field, $\operatorname{div}(\mathbf{curl}\, \mathbf{E}) = 0$.

37. **(C)**. Let $d(x, y) = C_{xx}C_{yy} - (C_{xy})^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Since $d(1, 1) > 0$ and $C_{xx}(1, 1) > 0$, $(1, 1, -1)$ is a relative minimum by the Second Partials Test.

38. **(C)**. The plane $y = 3x$ intersects the plane $y = 3$ when $x = 1$. If the order of integration is $dz\, dy\, dx$, the bounds are $0 \le z \le \sqrt{9 - y^2}$, $3x \le y \le 3$, and $0 \le x \le 1$ so the value of the triple integral is $\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx = 27/8$.

39. **(B)**. The distance from the origin to (x, y) is $\sqrt{x^2+y^2} = \rho(x, y)$ so the mass of the lamina is $\iint_R \rho(x, y)\, dA = \iint_R \sqrt{x^2+y^2}\, dA$. Switch to polar coordinates and obtain $\int_\pi^{3\pi/2} \int_0^5 r\,(r\, dr\, d\theta) = \int_\pi^{3\pi/2} \int_0^5 r^2\, dr\, d\theta = 125\pi/6$.

40. **(D)**. $W(t, te^t)(t) = \left|\begin{pmatrix} t & te^t \\ 1 & te^t + e^t \end{pmatrix}\right| = t^2e^t + te^t - te^t = t^2e^t \to W(t, te^t)(2) = 4e^2$.