Mu Alpha Theta National Convention: Denver, 2001

Logarithms and Exponents Topic Test Solutions – Theta Division

1) $3^{2x-1} = 27$

so if we take the log (base 3) of both sides, we get:

$2x - 1 = 3$

2x = 4

x = 2 , which is B.

2) Evaluate: $\log_2 16$

asks: To what power do we take 2 to get 16. By inspection, this is 4, which is A.

3) Which of the following is equivalent to $\frac{\ln 6}{\ln 2} - \log_2 6$?

(A) 0 (B) $\frac{\log_2 6}{2}$ (C) 1 (D) $\frac{3\log_2 6}{2}$ (E) NOTA

One of the common equality theorems about logarithms is $\frac{\log_c a}{\log_c b} = \log_b a$, so

$\frac{\ln 6}{\ln 2} = \frac{\log_e 6}{\log_e 2} = \log_2 6$, and thus $\frac{\ln 6}{\ln 2} - \log_2 6 = \log_2 6 - \log_2 6 = 0$, which is A.

4) Which of the following is equivalent to $\left(\sqrt[4]{x^3}\right)^{\frac{4}{6}}$?

(A) $\sqrt{x}$ (B) $-\sqrt{x}$ (C) $\frac{1}{\sqrt{x}}$ (D) $\sqrt{x^2}$ (E) NOTA

$\left(\sqrt[4]{x^3}\right)^{\frac{4}{6}} = \left(x^{\frac{3}{4}}\right)^{\frac{4}{6}} = x^{\frac{3}{4}\bullet\frac{4}{6}} = x^{\frac{3}{6}} = x^{\frac{1}{2}} = \sqrt{x}$, which is A.

5) Evaluate: $\log_3 36 - \log_3 12$

Use a common theorem for logarithms: $\log_a b - \log_a c = \log_a \frac{b}{c}$, so we get

$\log_3 36 - \log_3 12 = \log_3 \frac{36}{12} = \log_3 3 = 1$, which is B.

6) Evaluate: $\log_3 81$

$3^4 = 81$, so $\log_3 81 = 4$, which is C.

7) Solve for x: $\log_4(\log_3(\log_2 x)) = 0$

If $\log_a(y) = 0$, then $y = 1$, so $(\log_3(\log_2 x)) = 1$, which means that since $\log_a b = c \Rightarrow a^c = b$, $\log_2 x = 3$, and thus $x = 2^3 = 8$, which is B.

8) Solve for x: $\sqrt{x^2 + 6x + 9} = 4$

$4 = \sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2}$, so
$\pm(x+3) = 4$, and thus
$x + 3 = 4 \Rightarrow x = 1$, or $-(x+3) = 4 \Rightarrow x + 3 = -4 \Rightarrow x = -7$, and
–7 and 1 is B

9) Which of the following is equivalent to: $\frac{1}{2}\ln(16) + \ln(2) + \frac{1}{3}\ln(8^2)$?

(A) $4\ln 2$ (B) $\frac{5}{\log_2 e}$ (C) $\ln(2^3)$ (D) $\ln 16 - \ln 4$ (E) NOTA

$$\frac{1}{2}\ln(16) + \ln(2) + \frac{1}{3}\ln(8^2) = \ln\left(16^{\frac{1}{2}}\right) + \ln(2) + \ln\left((8^2)^{\frac{1}{3}}\right) = \ln(4) + \ln(2) + \ln\left(8^{\frac{2}{3}}\right) =$$
$$\ln\left(4 \bullet 2 \bullet \left(8^{\frac{2}{3}}\right)\right) = \ln(8 \bullet 2^2) = \ln(32) = \log_e 32 = \frac{\log_2 32}{\log_2 e} = \frac{5}{\log_2 e}$$
This is B.

10) Evaluate: $3^3 + 3^3 + 3^3$

$3^3 + 3^3 + 3^3 = 3(3^3) = 3^4$, which is A.

11) Evaluate: $\frac{1}{\log_3 24} + \frac{2}{\log_5 24} - \frac{1}{\log_{75} 24}$

$$\frac{\log_c a}{\log_c b} = \log_b a, so \frac{x}{\log_c b} = \frac{\log_c c^x}{\log_c b} = \log_b c^x, so$$
We know that $\frac{1}{\log_3 24} + \frac{2}{\log_5 24} - \frac{1}{\log_{75} 24} = \log_{24} 3 + \log_{24} 5^2 - \log_{24} 75 = \log_{24}\left(\frac{3 \bullet 25}{75}\right)$
$= \log_{24} 1 = 0$
This is D.

12) Simplify: $\dfrac{3^{2x}3^{1-x}9^{\frac{x}{2}}}{27^{\frac{2}{3}x-1}}$

(A) 3^{1-x} (B) 27 (C) 81 (D) 3^{2x+1}

$$\frac{3^{2x}3^{1-x}9^{\frac{x}{2}}}{27^{\frac{2}{3}x-1}}=\frac{3^{2x+1-x}\left(3^2\right)^{\frac{x}{2}}}{\left(3^3\right)^{\frac{2}{3}x-1}}=\frac{3^{x+1}\left(3^x\right)}{3^{2x-3}}=\frac{3^{2x+1}}{3^{2x-3}}=\frac{\left(3^{2x}\right)(3)}{\left(3^{2x}\right)\left(3^{-3}\right)}=\frac{3}{3^{-3}}=3^4=81,$$

which is C.

13) Which of the following is equivalent to $\log_8 xy^2-\dfrac{\frac{2}{3}}{\log_x\frac{1}{2}}$

(A) $\dfrac{2\log_8 xy}{3}$ (B) $\log_2\left(xy^{\frac{2}{3}}\right)$ (C) $\log_x\left(8y^2+2^{\frac{2}{3}}\right)$ (D) $6\log_2 xy$ (E) NOTA

By using the same trick used in problem 11, we quickly get that

$$\frac{\frac{2}{3}}{\log_x\frac{1}{2}}=\frac{\frac{2}{3}}{-\log_x 2}=-\log_2 x^{\frac{2}{3}}\text{, and}$$

$$\log_8 xy^2=\frac{\log_2 xy^2}{\log_2 8}=\frac{\log_2 xy^2}{3}=\log_2\left(\left(xy^2\right)^{\frac{1}{3}}\right)=\log_2 x^{\frac{1}{3}}y^{\frac{2}{3}}\text{, so}$$

$$\log_8 xy^2-\frac{\frac{2}{3}}{\log_x\frac{1}{2}}=\log_2 x^{\frac{1}{3}}y^{\frac{2}{3}}-\left(-\log_2 x^{\frac{2}{3}}\right)=\log_2\left(\left(x^{\frac{1}{3}}y^{\frac{2}{3}}\right)\left(x^{\frac{2}{3}}\right)\right)=\log_2\left(xy^{\frac{2}{3}}\right).$$

This is B.

14) Simplify: $\sqrt[5]{x^{10}}$

(A) x^2 (B) $-x^2$ (C) $-x^{\frac{1}{2}}$ (D) $x^{\frac{1}{2}}$ (E) NOTA

$\sqrt[5]{x^{10}}=x^{\frac{10}{5}}=x^2$, which is A. We have an odd powered root, so $-x^2$ does not work.

Mu Alpha Theta National Convention: Denver, 2001
Logarithms and Exponents Topic Test Solutions – Theta Division

15) If x and y are positive, then $\log_y 2x = \log_{2x} y =$

(A) $\log_y x$ (B) $\log_{2x} y^2$ (C) 2 (D) 1 (E) NOTA

if $\log_y 2x = \log_{2x} y$, then by the same trick as in problem 11, we get

$\log_y 2x = \log_{2x} y = \dfrac{1}{\log_y 2x}$, so $(\log_y 2x)^2 = 1$, and thus $(\log_y 2x) = \pm 1$.

Either of these solutions will work as $2x = y$ and $2x = \dfrac{1}{y}$ will both result with positive x and y values. Thus the answer will be E: 1 and -1.

16) Given that $\sqrt{\log_2 x} = \log_x 2$, solve for x, where $x > 0$.

(A) 1 (B) 2 (C) 4 (D) 8 (E) NOTA

$\sqrt{\log_2 x} = \log_x 2 = \dfrac{1}{\log_2 x}$, so $(\log_2 x)^{\frac{3}{2}} = (\sqrt{\log_2 x})(\log_2 x) = 1$, so $\log_2 x = 1$, and thus x=2, which is B.

17) Solve for z: $\log_{2z} 256 = 2$

(A) 25 (B) 5 (C) 2 (D) 8 (E) NOTA

$\log_{2z} 256 = 2$, so $4z^2 = (2z)^2 = 256$, and thus $z^2 = 64$.
Thus $z = \pm 8$. But since we are taking the log base 2z, we may rule out the negative case, giving us D.

18) Which of the following is equivalent to $a^{\ln b}$?

(A) $a^{b \ln a}$ (B) $e^{a \ln b}$ (C) $\ln(e^{\ln a})$ (D) $b^{\ln a}$ (E) NOTA

$a^{\ln b} = e^{\ln(a^{\ln b})} = e^{(\ln b)(\ln a)} = e^{\ln b^{\ln a}} = b^{\ln a}$, which is D.

19) Solve for x: $4^x = 8$

(A) $\dfrac{2}{3}$ (B) $\dfrac{5}{3}$ (C) $\dfrac{3}{2}$ (D) $\dfrac{5}{2}$ (E) NOTA

$4^x = 8$, so $x = \log_4 8 = \dfrac{3}{2}$, which is C.

20) What choices of B and C, <u>respectively</u>, would make the following equalities true? $\frac{\log C}{\log B} = \frac{C}{B} = \frac{3}{2}$.

(A) $\left(\frac{3}{2}\right)^2 and \left(\frac{3}{2}\right)^3$ (B) $\left(\frac{2}{3}\right)^4 and \left(\frac{2}{3}\right)^5$ (C) $\left(\frac{3}{2}\right)^4 and \left(\frac{3}{2}\right)^3$ (D) $\left(\frac{2}{3}\right)^2 and \left(\frac{2}{3}\right)^3$ (E) NOTA

$$\frac{\log C}{\log B} = \frac{\log\left(\frac{3}{2}\right)^3}{\log\left(\frac{3}{2}\right)^2} = \frac{3\log\frac{3}{2}}{2\log\frac{3}{2}} = \frac{3}{2} = \frac{\left(\frac{3}{2}\right)^3}{\left(\frac{3}{2}\right)^2} = \frac{C}{B}$$

21) Evaluate: $\log_{21} 7 + \log_{21} 3$

(A) $\log_{21} 10$ (B) $\frac{1}{2}$ (C) $\log_{21}\frac{7}{3}$ (D) 1 (E) NOTA

$\log_{21} 7 + \log_{21} 3 = \log_{21}(7 \bullet 3) = \log_{21} 21 = 1$, which is D.

22) If x is an integer such that $x > 1$, which of the following is always less than or equal to $\log_2(x!)$

(A) $x\log_2 x$ (B) $\log_2\sqrt{\left(\frac{x}{2}\right)^x}$ (C) x^2 (D) $\frac{x!}{\log_2 x}$ (E) NOTA

$4\log_2 4 = 4 \bullet 2 = 8 > \log_2 24 = \log_2 4!$, so A is out, while $2^2 = 4 > 1 = \log_2 2$, so C is out, and $\frac{2!}{\log_2 2} = 2 > 1 = \log_2 2!$. So D is out, too. $\log_2\sqrt{\left(\frac{x}{2}\right)^x} \le \log_2(x!) \Leftrightarrow \sqrt{\left(\frac{x}{2}\right)^x} \le x!$, but if we compare pair-wise the first $\frac{x}{2}$ multiples of $\sqrt{\left(\frac{x}{2}\right)^x}$ and $x!$, then we can think about $\left\{\left(\frac{x}{2}, x\right), \left(\frac{x}{2}, x-1\right), \ldots, \left(\frac{x}{2}, x - \left\lfloor \frac{x}{2} \right\rfloor\right)\right\}$, so the second term is always greater than or equal to the first, and thus $\sqrt{\left(\frac{x}{2}\right)^x} \le x!$ for all integers x>1. Thus we get B.

Mu Alpha Theta National Convention: Denver, 2001
Logarithms and Exponents Topic Test Solutions – Theta Division

23) Evaluate: $\log_8 32 - \log_8 16$

(A) $\frac{4}{3}$ (B) $\frac{1}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{4}$ (E) NOTA

$\log_8 32 - \log_8 16 = \log_8 \frac{32}{16} = \log_8 2 = \frac{1}{3}$, since 2 is the 3rd root of 8, so we get B.

24) If $\log_y x = \frac{2}{3}$, what is $\log_x y$?

(A) 1 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\log_x 2y$ (E) NOTA

$\frac{2}{3} = \log_y x = \frac{\log_c x}{\log_c y} = \frac{1}{\frac{\log_c y}{\log_c x}} = \frac{1}{\log_x y}$, so $\log_x y = \frac{3}{2}$, which is C.

25) Which of the following is equivalent to $\frac{\log_4 12}{\log_2 3}$?

(A) $\frac{\log_4 12}{1 - \log_3 4}$ (B) $\frac{1}{\log_2 3} + \frac{1}{2}$ (C) $\frac{1}{\log_2 3} + 1$ (D) $\log_2 \frac{144}{3}$ (E) NOTA

$\frac{\log_4 12}{\log_2 3} = \frac{\log_4 4 + \log_4 3}{\log_2 3} = \frac{1}{\log_2 3} + \frac{\log_4 3}{\log_2 3} = \frac{1}{\log_2 3} + \frac{\log_3 2}{\log_3 4} = \frac{1}{\log_2 3} + \log_4 2 = \frac{1}{\log_2 3} + \frac{1}{2}$,

which is B. Implicitly, twice we used the equality $\log_a b = \frac{1}{\log_b a}$, obtained by the following rearrangement: $\log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\frac{\log_c a}{\log_c b}} = \frac{1}{\log_b a}$.

26) Solve for y: $\frac{2}{3}\log_5 125 = y$

(A) 3 (B) 1 (C) 5 (D) 2 (E) NOTA

$y = \frac{2}{3}\log_5 125 = \frac{2}{3} \bullet 3 = 2$, which is D.

Mu Alpha Theta National Convention: Denver, 2001

Logarithms and Exponents Topic Test Solutions – Theta Division

27) Which of the following is equivalent to $a^{\ln\frac{d}{c}}$?

(A) $\left(\ln\frac{d}{c}\right)^a$ (B) $\frac{c^{\ln a}}{a^{\ln d}}$ (C) $\frac{d^{\ln a}}{a^{\ln c}}$ (D) $\ln\left|\left(\frac{d}{c}\right)^a\right|$ (E) NOTA

$a^{\ln\frac{d}{c}} = e^{\ln a^{\ln\frac{d}{c}}} = e^{\left(\ln\frac{d}{c}\right)\ln a} = e^{(\ln d - \ln c)\ln a} = e^{\ln a \ln d - \ln c \ln a} = e^{\ln d^{\ln a} - \ln c^{\ln a}} = \frac{e^{\ln d^{\ln a}}}{e^{\ln a^{\ln c}}} = \frac{d^{\ln a}}{a^{\ln c}}$, which is C.

28) If x and y are positive, then $\log x - \log y = \log\frac{y}{x} =$

(A) 0 (B) 1 (C) $\log_x y$ (D) $\log y^2$ (E) NOTA

$\log\frac{x}{y} = \log x - \log y = \log\frac{y}{x}$, but log is a one to one function, so $\frac{x}{y} = \frac{y}{x}$, so $x^2 = y^2$, but since both are positive, x=y. Thus $\log x - \log y = \log x - \log x = 0$, which is A.

29) Simplify: $\frac{\left(x^{\frac{5}{3}}\right)\left(x^{\frac{12}{5}}\right)}{\sqrt[15]{x}}$

(A) $\frac{x^4}{\sqrt[15]{x}} =$ (B) $x^{\frac{11}{15}}$ (C) x^4 (D) x^5 (E) NOTA

$\frac{\left(x^{\frac{5}{3}}\right)\left(x^{\frac{12}{5}}\right)}{\sqrt[15]{x}} = \frac{\left(x^{\frac{25}{15}}\right)\left(x^{\frac{36}{15}}\right)}{x^{\frac{1}{15}}} = x^{\frac{25+36-1}{15}} = x^4$, which is C.

30) Solve for x: $\log_x 5 + \log_x 125 = 4$

(A) 25 (B) 2 (C) 4 (D) 5 (E) NOTA

$4\log_x 5 = \log_x 5 + 3\log_x 5 = \log_x 5 + \log_x 5^3 = \log_x 5 + \log_x 125 = 4 \Leftrightarrow \log_x 5 = 1 \Leftrightarrow x = 5^1 = 5$, which is D.

31) Given that $18^{x^2+2x+4} = \left(54\sqrt{2}\right)^{x^2+4}$, solve for x.

(A) 2 (B) {4,2} (C) 3 (D) {5,3} (E) NOTA

$\left(\left(3\sqrt{2}\right)^2\right)^{x^2+2x+4} = 18^{x^2+2x+4} = \left(54\sqrt{2}\right)^{x^2+4} = \left(\left(3\sqrt{2}\right)^3\right)^{x^2+4}$, so we get

$\left(3\sqrt{2}\right)^{2x^2+4x+8} = \left(3\sqrt{2}\right)^{3x^2+12} \Rightarrow 2x^2+4x+8 = 3x^2+12 \Rightarrow x^2-4x+4=0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$

So we get A.

32) Solve for x: $\log_2(\log_2(\log_3 x)) = 1$

(A) 81 (B) 27 (C) 64 (D) 192 (E) NOTA

$\log_2(\log_2(\log_3 x)) = 1 \Rightarrow \log_2(\log_3 x) = 2 \Rightarrow \log_3 x = 2^2 = 4 \Rightarrow x = 3^4 = 81$, which is A.

33) Solve for x: $10c^{\log_{10} c^{x^2}} = m^{\log_{10} m + \log_m 10}$

(A) $\pm 10cm$ (B) $\pm\sqrt{\log_m c}$ (C) $\pm\log_c m$ (D) $\pm\log_m c$ (E) NOTA

$10c^{\log_{10} c^{x^2}} = m^{\log_{10} m+\log_m 10} \Leftrightarrow \log_m\left(10c^{\log_{10} c^{x^2}}\right) = \log_m\left(m^{\log_{10} m+\log_m 10}\right) \Leftrightarrow$

$\log_m 10 + x^2 \log_{10} c \log_m c = \log_m 10 + \log_{10} c^{x^2} \log_m c = \log_m 10 + \log_m c^{\log_{10} c^{x^2}} =$

$(\log_{10} m + \log_m 10)\log_m m = \log_{10} m + \log_m 10 \Leftrightarrow x^2 \log_{10} c \log_m c = \log_{10} m \Leftrightarrow$

$x^2 = \dfrac{\log_{10} m}{\log_{10} c \log_m c} = \dfrac{\log_c m}{\log_m c} = (\log_c m)^2 \Leftrightarrow x = \pm\log_c m$

This gives us C.

34) $x^y = y$, and $\log_3 y = z$. Solve for x with respect to z:

(A) $3^{\frac{z}{3^z}}$ (B) $\dfrac{z}{3^z}$ (C) $10^{\frac{z}{10^z}}$ (D) $\dfrac{z}{10^z}$ (E) NOTA

$\log_3 y = z \Rightarrow 3^z = y = x^y = x^{3^z} \Rightarrow z = \log_3 x^{3^z} = 3^z \log_3 x \Rightarrow \dfrac{z}{3^z} = \log_3 x \Rightarrow 3^{\frac{z}{3^z}} = x$, which gives us A.

35) If $x > 0$, which of the following is always less than $(4+x)^x$?

I. 4
II. 1
III. 4^{x+1}
IV. 2

(A) I, II, III, & IV (B) II & IV only
(C) II only (D) I,II, & III only (E) NOTA

$1 < 2 < 4 < 4^{x+1}$ for all x>0, so it is sufficient to show $1 < (4+x)^x$ for all x>0, and $2 \geq (4+x)^x$ for some x>0, to show that it is C. But $1 < (4+x)^x \Leftrightarrow 0 = \log_{4+x} 1 < \log_{4+x}(4+x)^x = x$, which is given. Additionally, we have
$2 < (4+x)^x \Leftrightarrow \log_{4+x} 2 < \log_{4+x}(4+x)^x = x$. Consider $x = 1$. Then $\log_{4+x} 2 = \log_5 2 < 1 = x$, so C is the correct choice.

36) Given that $x^{\log_x y} = x^x$, solve for y with respect to x.

(A) x (B) $1+x^3$ (C) $\left(\frac{1}{x}\right)^x$ (D) x^x (E) NOTA

$x^{\log_x y} = x^x \Rightarrow \log_x y = \log_x x^{\log_x y} = \log_x x^x = x \Rightarrow y = x^x$, which is D.

37) If a, b, and c are rational and $250^a 25^b 10^c = 10000$, evaluate $3a + 2b + c$.

(A) 3 (B) 4 (C) 1 (D) 5 (E) NOTA

$5^{3a+2b+c} 2^{a+c} = \left(5^3 \bullet 2^1\right)^a 5^{2b} (5 \bullet 2)^c 250^a 25^b 10^c = 10000 = 2^4 5^4$, so by unique factorization, we get $3a + 2b + c = 4$, which is B.

38) What is the sum of all the positive integral factors of 1280?

(A) 3293 (B) 2576 (C) 4346 (D) 3066 (E) NOTA

$1280 = 2^7 5$, so the sum we want is
$\sum_{i=0}^{7} 2^i + \sum_{i=0}^{7} 2^i 5 = 2^8 - 1 + 5\left(2^8 - 1\right) = 6\left(2^8 - 1\right) = 6(511) = 3066$, which is D.

39) How many real roots does the equation $x^4 = 16e^4$ have?

(A) 4 (B) 3 (C) 2 (D) 1 (E) NOTA

$x^4 = 16e^4 = (2e)^4 \Leftrightarrow x^2 = \pm(2e)^2 \Leftrightarrow x^2 = (2e)^2 \Leftrightarrow x = \pm 2e$, so there are 2 real roots, which is C.

40) Evaluate: $\sqrt{132+\sqrt{132+\sqrt{132+\cdots}}}$

(A) 11 (B) 6 (C) 12 (D) $\frac{91}{6}$ (E) NOTA

consider $x = \sqrt{132+\sqrt{132+\sqrt{132+\cdots}}}$, then $x^2 = 132+\sqrt{132+\sqrt{132+\sqrt{132+\cdots}}} = 132 + x$, so $(x-12)(x+11) = x^2 - x - 132 = 0$, but a negative solution does not make sense, so we get a single solution of 12, which is C.