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Number Theory Topic Test Solutions – Theta Division

1. **The easiest way to solve this problem is to find the total possible values of m because for each m there is exactly one possible n to pair with. The possible values of m consist of the total number of possible factors (positive and negative) of 30. The total number of positive factors of a number can be derived from that number's prime factorization. Add one to each of the exponents in the prime factorization (the total number of choices for the exponent of that prime in a factor) and multiply those numbers together (they are chosen independently from one another to identify each factor). The prime factorization of 30 has three primes (2, 3, 5) all raised to the first power. So there are (1+1)(1+1)(1+1) = 8 possible positive values for m, and also 8 negative values. For each m, there is exactly one n. There are 16 different ordered pairs.**

2. **48 = 2^4 x 3^1. Consider that the sum of all of the factors can be determined by:**

   ($2^4 + 2^3 + 2^2 + 2^1$ + 1)(3^1 + 1) because each and every factor is represented one and only once as one of the products of a power of each of the prime numbers in 48's prime factorization. Also notice that ($2^4 + 2^3 + 2^2 + 2^1$ + 1)(2-1) = (2^5-1). So a more compact formula can be derived to find the sums of factors of ANY integer. [Try to derive this formula completely as an exercise.]

The sum of all of its positive factors is thus ($2^5 - 1$)($3^2 - 1$)/[(2-1)(3-1)] = 124. Subtracting 48 (proper factors only) yields 76 as the answer.

3. **The trick is to find the number of powers of 10 which divide 134!. This means finding the exponents of 2 and 5 in the prime factorization of 134!. The exponent of 2 will clearly be greater than the exponent of 5, so we only need find the exponent of 5. 134/5 = 26 with a remainder. 26/5 = 5 with a remainder. 5/5 = 1. That means that there are 26 + 5 + 1 = 32 times in which 5 is included in the prime factorization of 134!.**

4. **The solution to a problem like this is not straightforward until the student learns a significant amount of number theory. The best way for a novice number theorist to tackle this problem is probably to think about how the number 12 can be created from the exponents in the prime factorization of a number. 12 = (3)(2)(2) = (4)(3) = (6)(2) = etc., and deriving exponents from these possibilities yields various results. Of course, the smallest prime in the final answer (2 is the smallest possible) should have the highest exponent or else there is a smaller solution. 12 = (3)(2)(2) gives exponents of 2, 1, and 1. Using the three smallest possible primes (2, 3, and 5), we find that 60 is indeed the smallest possible answer.**

5. **This problem is similar to the last except that the number 3 cannot be in the prime factorization. The resulting answer is 140, but be wary in that the creation of exceptions in problems like this CAN lead to a change in the way that the exponent choices get multiplied – (3)(2)(2) vs. (4)(3).**

6. 311_8 **= 3(**8^2**) + 8 + 1 = (2 + 1)(**2^6**) +** 2^3 **+ 1 =** $2^7 + 2^6 + 2^3$ **+ 1 =** 11001001_2**.**

7. 222,222 = 2 x 3 x 7 x 11 x 13 x 37. The answer is therefore 6.

8. 543_6 **= 5(36) + 4(6) + 3 = 207 =** $179_n = n^2$ **+ 7n + 9. Thus n = 11 or –18 and we ignore the negative solution.**

9. 331 is the only one of the answers which leaves a remainder of 1 when divided by 3.

10. 311/3 = 103 with a remainder. 103/3 = 34 with a remainder. 34/3 = 11 with a remainder. 11/3 = 3 with a remainder. 3/3 = 1. 103 + 34 + 11 + 3 + 1 = 152.

11. To find the answer, simply sum the first 110 counting numbers and subtract out the 10 which are perfect squares. The sum of the first 110 counting numbers is (110)(111)/2 = 6105. The sum of the first ten (non-zero) perfect squares is (10)(11)(21)/6 = 385. 6105 – 385 = 5720.

12. Call the number AB where A is the tens digit and B is the units digit.

BA – AB = 10(B - A) + (A - B) = 9(B - A) = 36. Thus B – A = 4.

13. 2001/9 = 222 remainder 3. 3 is the last digit. 222/9 = 24 remainder 6. 6 is the second to last digit. 24/9 = 2 remainder 6. 2 and 6 are the first two digits. 2 + 6 + 6 + 3 = 17.

14. 84 = 2^2 **x 3 x 7. The sum of the positive factors is thus**

$$[(2^3 - 1)(3^2 - 1)(7^2 - 1)]/[(2 - 1)(3 - 1)(7 - 1)] = 224$$

15. N ≡ 1 (mod 2), so we can let N = (2M – 1) where M is a positive integer. Thus we know that 2M-1 ≡ 2 (mod 3) hence 2M ≡ 0 (mod 3) and thus M ≡ 0 (mod 3). We can let M = 3P where P is a positive integer. Now we can determine N (mod 6). N = 2M – 1 = 6P – 1 ≡ -1 (mod 6) ≡ 5 (mod 6).

16. Let the number be N and define an integer m such that N = 13m.

Since N ≡ 1 (mod 7), we can say 13m ≡ 1 (mod 7) and thus 14m – 13m ≡ 0 - 1 (mod 7).

Thus m ≡ 6 (mod 7). The smallest positive m is 6. Thus the smallest N = 13(6) = 78.

17. 221 = (13)(17); 1001 = (7)(11)(13); 1728 = (2^6)(3^3); 2737 = (7)(17)(23). Clearly 1728 is the answer as it contains no factors common to the other three.

18. The sum is equivalent to $\sum_{n=1}^{100} 4n-2$ = ($4\sum_{n=1}^{100} n$) – 2(100) = 4(100)(101)/2 – 200 = 20,000.

19. 10a ≡ 1 (mod 13) and 13a ≡ 0 (mod 13).

17a = 3(10a) – 13a ≡ 3(1) – 1(0) (mod 13) ≡ 3 (mod 13).

20. 6A6B ≡ 0 (mod 9); and 6A6B ≡ 0 (mod 8). A nice trick that most students should know (and work to discover why the relationship works) is that a decimal number is a multiple of 9 when and only when the sum of its digits is a multiple of 9. Thus we know that 12 + A + B ≡ 3 + A + B ≡ 0 (mod 9). We also know that 6000 ≡ 0 (mod 8) and thus 6A6B ≡ 6A6B – 6000 ≡ A6B (mod 8). B must of course be even. Now the student can plug in the small range of possible A's and add them. 2 + 7 = 9.

21. 2 x 7 x 17 x 37 = 8,806

22. Given that 3x ≡ 4 (mod 5), we can say that 3x ≡ 4+5 (mod 5), then 3x ≡ 9 (mod 5), and thus x ≡ 3 (mod 5). Likewise we can find that x ≡ 4 (mod 7). From the latter of these relationships, we can say that x = 7y –3 for any positive integer y. Thus we know that 7y -3 ≡ 3 (mod 5) => 7y ≡ 1 (mod 5) ≡ 21 (mod 5). Thus y ≡ 3 (mod 5). We can say that y = 5z –2 for any positive integers z. From the relationship between x and y, we now know that x = 7(5z –2) –3 = 35z – 17. Thus x ≡ -17 (mod 35) ≡ 18 (mod 35). 53 is the only answer which satisfies this relationship.

23. There are several nice ways of doing this problem. The solution can be easily obtained by merely adding all of the factors and then subtracting all of the factors of its largest odd factor (27). Using the formula from problem 2, the sum of all the factors is 5080. The sum of the factors of 27 is 40. 5080 – 40 = 5040.

24. The trick is to find the LCM of (2-10) and subtract 1. The LCM of (2-10) will have a prime factorization of 2^3 x 3^2 x 5 x 7 = 2520. 2520 – 1 = 2519.

25. The sum of the first n perfect cubes is $[(n)(n+1)/2]^2$. Plugging in yields (10)(11)/2 = 55. 55 X 55 = 3025.

26. We know that n(n + 1) ≡ 0 (mod 3) and that n(n + 1) ≡ 0 (mod 61). 60 is the smallest positive n which satisfies the second relation and it also happens to satisfy the first, so we look no further.

27. 3 x 5 = 15. 5 x 7 = 35. 11 x 13 = 143. 17 x 19 = 323. 29 x 31 = 899. 41 x 43 = 1763. 1763 + 899 + 323 + 143 + 35 + 15 = 3,178.

28. The Pigeonhole principle says that there are at least two killers with 13 kills or at least one with 14. Since Pat B. killed MORE aliens than any other killer did, he must have killed at least 14 aliens.

29. $7^4 \equiv$ **1 (mod 100). Thus** $7^{707} \equiv 7^{4(176)+3} \equiv (1^{176})(7^3) \equiv 7^3 \equiv$ **343** $\equiv$ **43 (mod 100). Thus the answer is 4.**

30. 329 x 999 = 32571. A nice test for divisibility of a number by 99 is to add the digits pairwise starting from the right 71 + 25 + 3 = 99. A number is a multiple of 99 if and only if its related pairwise digit sum is a multiple of 99.

31. 9984 = 10,000 – 16 = (100 + 4)(100 – 4). The difference is 8.

32. 297 = 3^3 **x 11; 481 = 13 x 37; and 672 =** 2^5 **x 3 x 7. The LCM =** 2^5 **x** 3^3 **x 7 x 11 x 13 x 37.**

The LCM = 31,999,968.

33. 40 has 4 x 2 = 8 positive divisors. For each divisor d, there is exactly one other divisor m such that dm = 40. Thus there are 4 pairs of divisors, the product of each pair being 40. Thus the product of all the divisors is 40^4**.**

34. The sum of the first N positive perfect squares is (N)(N+1)(2N+1)/6. The smallest N which can make that numerator a multiple of 41 is N = 20.

35. $5^2 \equiv$ **1 (mod 8), thus** $5^{2(150)+1} \equiv$ **(**1^{150}**)(5) (mod 8)** $\equiv$ **5 (mod 8).**

36. When a number is expressed in base 10, it is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The same is true for multiples of 4 expressed in base 5 (for the same reason – a good exercise is to figure out exactly why this is true). Given that fact, we are looking for the number 1111_5**. 125 + 25 + 5 + 1 = 156, which is indeed a multiple of 4.**

37. In the prime factorization of M, the exponent of 2 can only be 1. In the prime factorization of M, the exponent of 2 can only be 3. MN must therefore contain 2 with an exponent of 4. But MN is not divisible by 32, so MN divided by 32 must leave a remainder of 16.

38. How many ways can we make such a number with 9 factors? With 8, 7, less? We use the formula for the number of positive factors and realize that we are just ordering the exponents (plus one) of the primes 2 and 3 to produce the desired number of factors.

9 = (9)(1) = (3)(3) = (1)(9); 8 = (8)(1) = (4)(2) = (2)(4) = (1)(8); 7 = (7)(1) = (1)(7); 6 = (6)(1) = (3)(2) = (2)(3) = (1)(6); etc. If it is not clear to the reader how to construct each number that we are looking for, find one of the numbers with 8 factors. 8 = (4)(2). If 4 and 2 represent one greater than each of the exponents associated with the primes 2 and 3, then we have a prime factorization of $(2^3)(3^1)$ **= 24.**

39. M = 2^4 **x** 3^2 **x 5 x 7 x 11 x 13 x 17 x 19. The number of positive factors is thus 5 x 3 x 2 x 2 x 2 x 2 x 2 x 2 = 960.**

40. **This problem can be tediously worked out by long division. There is a much simpler way however. Find the prime factorization of the large number.**

337,500,000 = $(2^5)(3^3)(5^8)$**. Divide both this number and 128 by 32. Now take the remaining portion of the large number and find its remainder when divided by 4 (which is 128/32):** $(3^3)(5^8) \equiv (-1)^3(1^8)$ **(mod 4)** $\equiv$ **-1 (mod 4)** $\equiv$ **3 (mod 4). Multiplying both sides back by 32 tells the solver that the original number is congruent to 96 (mod 128).**