Integration Test Solutions

1. If f is a continuous even function, and $\int_0^a f(x)dx = 3$, then $\int_{-a}^a 3f(x)dx =$

(E) NOTA

Since f is even, $\int_0^a f(x)\,dx = \int_{-a}^0 f(x)dx = 3$

$$\int_{-a}^a 3f(x)dx = 3\int_{-a}^a f(x)dx = 3\left[\int_{-a}^0 f(x)dx + \int_0^a f(x)dx\right] = 3(3+3) = 18$$

2. Approximate the value of $\int_1^4 \frac{1}{x}dx$ by the trapezoidal rule using n=3.

(C) 1.458

$$\int_1^4 \frac{1}{x}\,dx \approx \left(\frac{4-1}{3}\right)\left(\frac{1}{2}\right)(1)\left(1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{35}{12}\right) \approx 1.458$$

3. Find the average value of the function $f(x) = \frac{x^2+4}{x}$ over the interval [1,4].

(A) $\frac{5}{2} + \frac{8}{3}\ln 2$

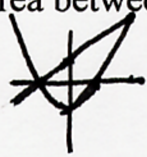
$$A_v = \frac{1}{4-1}\int_1^4 \frac{x^2+4}{x}dx = \frac{1}{3}\int_1^4\left(x + \frac{4}{x}\right)dx = \frac{1}{3}\left(\frac{x^2}{2} + 4\ln x\right)\Big]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{2} + 4\ln 4 - \frac{1}{2} - 4\ln 1\right] = \frac{1}{3}\left(8 + 4\ln 4 - \frac{1}{2}\right) = \frac{1}{3}\left(7\tfrac{1}{2} + 4\ln 4\right)$$

$$= \frac{15}{6} + \frac{4}{3}\ln 4 = \frac{5}{2} + \frac{4}{3}\ln 2^2 = \frac{5}{2} + \frac{8}{3}\ln 2$$

4. Find the area between the curves $y = x+1$ and $y = x^2 - 1$.

(B) $\frac{9}{2}$



$x+1 = x^2-1$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = -1, 2$

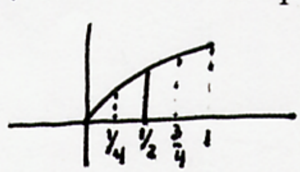
$$\text{Area} = \int_{-1}^2 [x+1-(x^2-1)]dx = \int_{-1}^2(-x^2+x+2)dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x\Big]_{-1}^2 = -\frac{8}{3} + \frac{4}{2} + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= -\frac{8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2 = 4\tfrac{1}{2} \text{ or } \frac{9}{2}$$

5. Approximate the value of $\int_0^1 \sqrt{x}dx$ to 3 decimal places using the Midpoint Rule, with n=2.

(A) 0.683



$f\left(\frac{1}{4}\right) = \frac{1}{2}$

$f\left(\frac{3}{4}\right) = \frac{\sqrt{3}}{2}$

$$A \approx \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{\sqrt{3}}{4} = \frac{1+\sqrt{3}}{4} \approx \frac{2.732}{4} = 0.683$$

6. If $F(x) = \int_0^{x^2} \frac{1}{3t^2 - 1}\,dt$, find $F'(x)$.

(E) NOTA

By the second fundamental theorem of calculus,

$$\frac{d}{dx}\left(\int_0^{x^2} \frac{1}{3t^2-1}\,dt\right) = \frac{1}{3(x^2)^2-1} \cdot \frac{d}{dx}(x^2) = \frac{2x}{3x^4-1}$$

7. Find $\int \frac{2}{9+x^2}\,dx$.

(B) $\frac{2}{3}\arctan\frac{x}{3} + C$

$$\int \frac{2}{9+x^2}\,dx = 2\int \frac{1}{9+x^2}\,dx \qquad 9 = a^2 \quad 3 = a$$

$$2\int \frac{1}{a^2+x^2}\,dx = \frac{2}{a}\arctan\frac{x}{a} + C = \frac{2}{3}\arctan\frac{x}{3} + C$$

8. $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx =$

(B) $1 - \frac{\pi}{4}$

$$\tan^2 x = \sec^2 x - 1$$

$$\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx = \int_0^{\frac{\pi}{4}} \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} 1\,dx =$$

$$\tan x - x\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0) = 1 - \frac{\pi}{4}$$

9. $\int \frac{1}{1+e^x}\,dx =$

(A) $x - \ln(1+e^x) + C$

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int\left(1 - \frac{e^x}{1+e^x}\right)dx$$

let $u = 1+e^x \quad du = e^x\,dx$

$$\int 1\,dx - \int \frac{du}{u} = x - \ln|u| + C = x - \ln(1+e^x) + C$$

note: $1+e^x > 0$ for all x, so absolute value is unnecessary.

10. Find $\int_1^3 \frac{dx}{2x^2 - 8x + 10}$.

(C) $\frac{\pi}{4}$

$$\int_1^3 \frac{dx}{2x^2-8x+10} = \frac{1}{2}\int_1^3 \frac{dx}{x^2-4x+5} = \frac{1}{2}\int_1^3 \frac{dx}{(x^2-4x+4)+1}$$

$$= \frac{1}{2}\int_1^3 \frac{dx}{(x-2)^2+1} = \frac{1}{2}\arctan(x-2)\Big]_1^3$$

$$= \frac{1}{2}\arctan(3-2) - \frac{1}{2}\arctan(1-2)$$

$$= \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan(-1) = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\left(\frac{-\pi}{4}\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$$

11. If $y' = \dfrac{2x}{x^2-8}$ and if the point (3,1) is on the graph of y, find the value of y when $x = 2$.

$\dfrac{dy}{dx} = \dfrac{2x}{x^2-8}$ $\quad dy = \dfrac{2x\,dx}{x^2-8}$ $\quad y = \int \dfrac{2x\,dx}{x^2-8}$

(D) $1+2\ln 2$

$u = x^2-8$ $\quad du = 2x\,dx$ $\quad y = \int \dfrac{du}{u} = \ln|u| + C$

$= \ln|x^2-8| + C$ $\quad 1 = \ln(3^2-8)+C$

$1 = \ln(1)+C$ $\quad$ so $C = 1$

$y = \ln|x^2-8|+1$ so $\ln|2^2-8|+1 =$

$\ln 4 + 1 = 2\ln 2 + 1$

12. Find $\lim\limits_{n\to\infty} \dfrac{3}{n}\sum\limits_{i=1}^{n} \left(3(1+\tfrac{3i}{n})^2 - 2(1+\tfrac{3i}{n})\right)$

This sum is equivalent to $\int_1^4 (3x^2-2x)\,dx$

(D) 48

Since the function begins at 1, and $b-a=3$.

$\int_1^4 (3x^2-2x)\,dx = x^3 - x^2\Big]_1^4 = 64-16-(1-1) = 48$

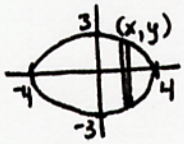
13. $\displaystyle\int_0^4 (16-x^2)^{\frac{1}{2}}\,dx =$

$y = \sqrt{16-x^2}$ is a semicircle of radius 4, centered at (0,0).
Since the limits of integration are 0 and 4, the area is the quarter circle. Area of $\odot = \pi(4)^2 = 16\pi$.

(C) 4π

Hence, the quarter circle has area $\dfrac{16\pi}{4} = 4\pi$

14. Find the volume of the solid whose base is the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ and whose cross sections, taken perpendicular to the x-axis are all squares.



side of 1 square $= 2\cdot y = 2\sqrt{9-\frac{9x^2}{16}}$

(D) 192

Area of 1 square $= 4\left(9-\frac{9x^2}{16}\right) = 36 - \frac{9x^2}{4}$

Volume $= \displaystyle\int_{-4}^{4}\left(36-\frac{9x^2}{4}\right)dx = 2\int_0^4\left(36-\frac{9x^2}{4}\right)dx = 2\left(36x - \frac{3x^3}{4}\right]_0^4$

$= 2(144-48) = 192$

15. Which one of the following four functions is not integrable over [1,5]?

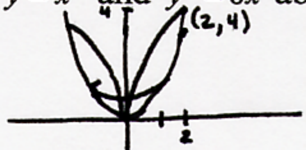
All functions which are continuous over [a,b] are integrable over [a,b], so (A), (B), (C) are integrable.

(E) NOTA

(D) is also integrable since the function is defined over [1,5] and the

$\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f(c_i)\,\Delta x_i$ exists.

16. Which one of the following four integrals can be used to determine the volume of a solid formed by revolving the region bounded by the curves $y = x^2$ and $y^2 = 8x$ about the y-axis?



(D) $\pi \int_0^4 (y - (\frac{1}{8}y^2)^2)\, dy$

using discs (washers),

$A = \pi \int_0^4 (\sqrt{y})^2 - (\frac{1}{8}y^2)^2\, dy$

$y = x^2 \Rightarrow x = \sqrt{y}$ $\quad 8x = y^2 \Rightarrow x = \frac{y^2}{8} = \frac{1}{8}y^2$

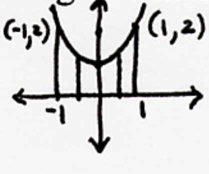
17. $\int_0^4 \frac{x^2 + 3x + 2}{x + 2}\, dx =$

(A) 12

$\int_0^4 \frac{(x+2)(x+1)}{x+2}\, dx = \int_0^4 (x+1)\, dx = \frac{x^2}{2} + x \Big]_0^4$

$= \frac{16}{2} + 4 - 0 = 8 + 4 = 12$

18. Use inscribed rectangles with n=4 to estimate $\int_{-1}^{1} (x^2 + 1)\, dx$.



(B) $2\frac{1}{4}$

4 rectangles, so width is $\frac{1 - -1}{4} = \frac{2}{4} = \frac{1}{2}$

$A = \frac{1}{2} \cdot (f(-\frac{1}{2})) + \frac{1}{2}(f(0)) + \frac{1}{2}(f(0)) + \frac{1}{2}(f(\frac{1}{2}))$

$= \frac{1}{2}(\frac{5}{4}) + \frac{1}{2}(1) + \frac{1}{2}(1) + \frac{1}{2}(\frac{5}{4})$

$= \frac{1}{2}(\frac{9}{2}) = \frac{9}{4} = 2\frac{1}{4}$

19. $\int_0^1 \frac{3^x}{3^x + 1}\, dx =$

(A) $\frac{\ln 2}{\ln 3}$

$u = 3^x + 1 \Rightarrow du = 3^x \cdot \ln 3\, dx$ $\quad \frac{du}{\ln 3} = 3^x\, dx$

$u(0) = 2$

$u(1) = 4$

$\int_2^4 \frac{du}{\ln 3 \cdot u} = \frac{1}{\ln 3} \int_2^4 \frac{du}{u} = \frac{1}{\ln 3} (\ln u \Big]_2^4 = \frac{\ln 4 - \ln 2}{\ln 3}$

$= \frac{\ln 4 - \ln 2}{\ln 3} = \frac{2\ln 2 - \ln 2}{\ln 3} = \frac{\ln 2}{\ln 3}$

20. $\int \frac{1}{x^2 + 2x - 8}\, dx =$

(B) $\frac{1}{6} \ln \left| \frac{x-2}{x+4} \right| + C$

Integration using partial fractions.

$\frac{1}{x^2 + 2x - 8} = \frac{A}{(x+4)} + \frac{B}{(x-2)} = \frac{A(x-2) + B(x+4)}{(x+4)(x-2)}$

$Ax + Bx = 0$ $\quad -2A + 4B = 1$

$A = -B$

$-2(-B) + 4B = 6B = 1 \therefore B = \frac{1}{6}$ and $A = \frac{-1}{6}$

$\int \frac{dx}{x^2 + 2x - 8} = \int \left[\frac{-1/6}{(x+4)} + \frac{1/6}{(x-2)}\right] dx = \frac{1}{6} \int (\frac{1}{x-2} - \frac{1}{x+4})\, dx$

$= \frac{1}{6}(\ln|x-2| - \ln|x+4|) + C = \frac{1}{6} \ln \left| \frac{x-2}{x+4} \right| + C$

21. An object begins at rest at the point (5,0) and moves along the x-axis with constant acceleration. If the velocity of the object after 3 seconds is 12 units per second, where will the object be located at 10 seconds?

(C) (205,0)

$a = k$ so $v = kt + C_1$
since $v(0) = 0$, $C_1 = 0$ $\quad v = kt$
since $v(3) = 12$ $\quad k = 4$ $\quad v = 4t$
$s(t) = \int 4t\,dt = 2t^2 + C_2$
$s(0) = 5 \Rightarrow C_2 = 5$ $\quad s(t) = 2t^2 + 5$
so $s(10) = 2(10)^2 + 5 = 205$

22. Find the Riemann sum for $f(x) = x^2 - x$ over the interval [0, 6] using $x_0 = 0, x_1 = 3, x_2 = 4, x_3 = 6$ and $c_1 = 0, c_2 = 4, c_3 = 5$.

(C) 52

$R = f(0)(3) + f(4)(1) + f(5)\cdot 2$
$= 0\cdot 3 + 12\cdot 1 + 20\cdot 2 = 12 + 40 = 52$

23. If $\int_{-2}^{1} f(x)\,dx = 0$ and $\int_{0}^{1} f(x)\,dx = 4$, then $\int_{-2}^{0} (f(x)+2)\,dx =$

(C) 0

$\int_{-2}^{0} f(x)\,dx = \int_{-2}^{1} f(x)\,dx - \int_{0}^{1} f(x)\,dx = -4$
$\int_{-2}^{0} (f(x)+2)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{-2}^{0} 2\,dx = -4 + \left[2x\right]_{-2}^{0} = -4 + (0 - -4) = -4+4=0$

24. $\int x\sqrt{1-x}\,dx =$

(D) $-\frac{2}{15}(1-x)^{\frac{3}{2}}(3x+2)+C$

$u = 1-x$ so $x = 1-u$ and $du = -dx$
$\int x(1-x)^{1/2}\,du = -\int (1-u)\,u^{1/2}\cdot du$
$= \int (u-1)\,u^{1/2}\,du = \int u^{3/2} - u^{1/2}\,du =$
$\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-5) + C$
$= \frac{2}{15}(1-x)^{3/2}(3(1-x)-5) + C =$
$\frac{2}{15}(1-x)^{3/2}(-3x-2) + C$

25. Find $F'(x)$ given that $F(x) = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \tan^4 t\,dt$.

(B) 0

Since $\int_{\pi/3}^{\pi/4} \tan^4 t\,dt$ is a constant, its derivative $= 0$

26. Find the average value of the function $f(x)=\frac{1}{\sqrt{1-x^2}}$ from 0 to $\frac{1}{2}$.

(E) NOTA

$$A_v = \frac{1}{\frac{1}{2}-0}\int_0^{1/2}\frac{1}{\sqrt{1-x^2}}dx = 2\int_0^{1/2}\frac{dx}{\sqrt{1-x^2}} = 2\left[\arcsin x\right]_0^{1/2}$$
$$= 2\arcsin\tfrac{1}{2} - 2\arcsin 0 = 2\cdot\tfrac{\pi}{6} - 0 = \tfrac{\pi}{3}$$

27. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin^2 3x\cos 3x\, dx=$

(A) $\frac{-\sqrt{2}}{36}$

let $u=\sin 3x \Rightarrow du = 3\cos 3x\,dx \Rightarrow \frac{1}{3}du = \cos 3x\,dx$

$$\frac{1}{3}\int_{x=\pi/4}^{x=\pi/3} u^2\,du = \frac{1}{3}\cdot\frac{u^3}{3}\Big]_{\sqrt{2}/2}^{0} = 0 - \frac{1}{9}\left(\frac{\sqrt{2}}{2}\right)^3 = -\frac{1}{9}\cdot\frac{2\sqrt{2}}{8} = \frac{-\sqrt{2}}{36}$$

if $x=\pi/4$, $u=\sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$
if $x=\pi/3$, $u=\sin\pi=0$

28. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{3x^3}{1+x^2}$ given the initial condition $y(0)=1$.

(B) $\frac{3}{2}\left(x^2-\ln\left(1+x^2\right)\right)+1$

$\frac{3x^3}{1+x^2}$

$$\begin{array}{r} 3x - \frac{3x}{1+x^2} \\ 1+x^2 \overline{)\,3x^3} \\ 3x^3+3x \\ \hline -3x \end{array}$$

$$y = \int\left(3x - \frac{3x}{1+x^2}\right)dx =$$

$$\int 3x\,dx - \int\frac{3x\,dx}{1+x^2}$$ $u=1+x^2$, $du=2x\,dx$, $\frac{3}{2}du = 3x\,dx$

$$\frac{3x^2}{2} - \frac{3}{2}\int\frac{du}{u} = \frac{3x^2}{2} - \frac{3}{2}\ln(1+x^2) + C$$

and $y(0) = \frac{3\cdot 0}{2} - \frac{3}{2}\ln(1+0) + C = \quad C=1$

29. $\int_1^2 (5^3-3^2)\,dx=$

(C) 116

$5^3 - 3^2 = 125 - 9 = 116$

$$\int_1^2 116\,dx = 116x\Big]_1^2 = 116(2-1) = 116$$

30. $\int_0^1 10^{3x}\,dx=$

(D) $\frac{333}{\ln 10}$

$$\int_0^1 10^{3x}\,dx$$ let $u=3x$, $du=3dx$; $u(0)=0$, $u(1)=3$

$$\frac{1}{3}\int_0^3 10^u\,du = \frac{1}{3}\cdot\frac{10^u}{\ln 10}\Big]_0^3 = \frac{1}{3}\left(\frac{1000}{\ln 10} - \frac{1}{\ln 10}\right) =$$

$$\frac{1}{3}\left(\frac{999}{\ln 10}\right) = \frac{333}{\ln 10}$$