

Interscholar Test
FAMAT State Convention 2002

1. What are the next two terms in the following sequence?

10, 15, 23, 31, 41, 49, 59, 71, ...

2. A contestant on the Price is Right is playing the dice game to try and win a car. She has no idea what the actual price of the car is, and, unfortunately for her, none of the numbers she rolls is the exact digit in the price of the car. The first digit in the price of the car is given as a 1. For the second and third digits she rolls a 2 and a 3, respectively, and says that both the second and third digits in the price of the car are higher than their respective roll. For the fourth and fifth digits she rolls a 5 and a 6, respectively, and says that both the fourth and fifth digits are lower than their respective roll. What is the probability that she will get all four choices correct and win the car?

3. What is the ratio of the length to the width of an American flag?

4. If the time and day in Sydney, Australia is 8 AM Apr. 20, 2002, then what is the time and day at Grenelafe?

5. The lengths of the sides of a triangle form a harmonic progression. If the length of the shortest side of the triangle is 3, and the perimeter is 10, then what is the area of the triangle?

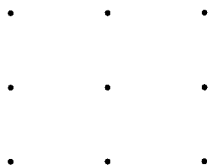
6. Frugal Frank is always interested in making money. One February he sat down and realized that he would only get four paychecks that year (because he gets paid every Friday). What is the next year that he will have five pay days in February?

7. A student wants to find the root closest in value to 2 in the following equation:

$$9x^4 - 76x^2 + 147 = 0.$$

He uses Newton's method with an initial guess of 2. What is the fewest number of iterations of Newton's method that will get an answer that is within ± 0.0001 of the correct answer?

8. What is the fewest number of line segments you can draw in the figure below without lifting your pencil and without retracing a line such that you will connect all 9 dots below (please draw your answer)?



9. A circle has a radius of 10. A 20-sided regular polygon circumscribes this circle. Another 20-sided regular polygon is inscribed in this circle. The ratio of the perimeter of the circumscribed polygon to the perimeter of the inscribed polygon can be written in the form $\sec N^\circ$, where N is the number of degrees between 0 and 360. What is the value of N to the nearest hundredth of a degree?

10. What is the sum of the following infinite series?

$$9 + 6 + 3 + 4/3 + 45/81 + \dots$$

11. How many dominoes are in a set of double 12s?

12. Evaluate the following definite integral:

$$\int_2^4 \frac{7x^2 - x - 66}{2x^3 - 9x^2 - 8x + 15} dx$$

13. Which of the following numbers are algebraic numbers (please list the letters on your answer sheet)?

A. 5 B. $5i$ C. e D. $\sqrt{3} + \sqrt{2}$ E. π F. $\sqrt[4]{3} - 2i$

14. Five horses are running in a horse race. Horse 1 has 2:1 odds of winning the race (which actually means that two times he will lose compared to every 1 time he wins); similarly, horse 2 has 3:1 odds of winning; and horses 3 and 4 each have 5:1 odds of winning. What are the odds that the horse 5 will win the race (please give the odds in the same order the other horse's odds were given)?

15. What is the maximum possible score in a single round of Yahtzee (round, not a match)?

16. A doctor tells Frugal Frank to take a certain long-term medicine twice a day at even intervals (in other words, every 12 hours). Frank likes to wake up every morning at 6:30 and go to sleep every night at 10:30. He decides to stretch his medication out a little to save money. He knows that if he takes the medication only once a day (every 24 hours), he will not benefit from it. What time interval (between 12 and 24 hours) will allow him to take his medication and not ever have to wake up in the middle of his nightly sleep?

17. A special plane curve, named after an astronomer, is described as a figure consisting of all those points for which the product of their distances to two fixed points (called the foci, separated by a distance $2a$) is a constant (b^2). The polar equation for this curve about the origin is given by the formula:

$$r^4 + a^4 - 2a^2 r^2 \cos(2\theta) = b^4$$

What is the name of this curve?

18. A piece of music has 96 measures in it, with each measure under a $4/4$ time signature. If the tempo throughout the music is set as $\text{♩} = 90$, then how many seconds will the music last?

19. Physicists use units known as Joules in order to measure work. How many Joules (to the nearest hundredth) are in 1000 pound-foot squared per second squared?

20. The fraction $8/31$ can be expressed as the sum of three unit fractions with different natural number denominators, A , B , and C (that is, $8/31 = 1/A + 1/B + 1/C$). What are the denominators A , B , and C if the result $A+B+C$ must be the minimum possible value?

21. Triangle MAT has sides of length m , a , and t and angles opposite those sides of M , A , and T , respectively. If m is 28 and a is 11, then what is the value of the expression below (give answer as a fraction of the form $\pm n/d$, where n and d are the smallest possible positive integers that correctly answer the question)?

$$\tan[0.5(M+A)]/\tan[0.5(M-A)]$$

22. What is the most points you can win in a row in tennis and still only win two games at some point during that streak (U.S. open rules, and winning a set tiebreaker counts as winning one game)?

23. What is the value of the following infinite product?

$$(2/1)(2/3)(4/3)(4/5)(6/5)(6/7) \dots$$

24. A meteorologist keeps her own record of the temperature at her home. She has observed the temperature every hour on the hour over a long period of time. When she created a plot of temperature over time, she noticed that it was very messy. She wants to create a smoother graph of the data, so she uses a Hanning filter. This filter weights data such that for every three points of data, the center point is given 1/2 weight and the outer two points are given 1/4 weight. For example, a part of her time series looks like this:

... 70 75 80 85 83 78 72 ...

After doing one Hanning filter through the data, she gets:

... 75 80 83.25 82.25 77.75 ...

Running another Hanning filter over this data would produce an even smoother result:

... 79.56 82.19 81.38 ...

For each Hanning filter used, the weight given to the center point in the data will be less and less. What is the value of this weight if she decides to run 6 Hanning filters over the data (assume that the center point is somewhere well in the middle of the data)?

25. Letters have been substituted for numbers in this multiplication problem. Determine the value of each letter. Then arrange the letters in order from 0 to 9 to reveal a word or phrase, and put this word or phrase on your answer sheet. No letter represents two different numbers in the problem.

$$\begin{array}{r} \\ \\ \\ \hline \\ \\ \\ \hline \end{array}$$

26. The following fortran program is run and the printed output is 21012.06250. What is the input?

```
PROGRAM TEST
real sum, input, output
integer i
sum=0.
input=????
do 10 i=1,10
    sum=sum+input*input*i*i
    input=input+1
10 continue
output=sum
write (*,20) output
20 format (f11.5)
stop
end
```

27. Eight numbers from 1 to 50 have a certain property about them. Seven of the numbers are (in no particular order): 28, 32, 12, 20, 50, 45, and 18. What is the other number?

28. A radar tower is 50 meters above the surface of the earth. It sends out a signal at an angle of 0.5 degrees above horizontal. A blip is noticed on the radar screen at a signal distance of 150 km. How high above the earth's surface (to the nearest meter) would this signal be (assuming a spherical earth of radius 6370 km)?

29. What is the spectral radius of the matrix $\begin{bmatrix} 4 & 21 & -5 \\ 0 & 0 & 1 \\ 2 & 3 & 6 \end{bmatrix}$?

30. How many zeros does 2002! end in?

31. In football, some of the designed passing plays are called timing routes. This is where the quarterback waits a certain amount of time before throwing the ball, and then, hopefully, his receiver will catch the ball while running at full speed. Here's an example of one play, a post pattern. The quarterback lines up in the shotgun 5 meters behind the line of scrimmage. A wide receiver lines up on the same yard line 20 meters to the right of the quarterback. The quarterback stays in the same spot on the field throughout the play. The receiver, however, accelerates straight down the field at a rate of $8-3t$ m s⁻². His speed rises at this rate until he reaches his peak speed, at which point he maintains the peak speed. At t equal to 2 seconds, the receiver changes his direction of motion to a path that is 30 degrees to the left of his previous path. The quarterback throws the ball at a horizontal velocity of 30 m s⁻¹ and a vertical velocity of 8 m s⁻¹. If you consider that only gravity affects the ball ($g=-9.80$ m s⁻²), that the field is flat, and that the quarterback throws the ball at the same level that the receiver catches the ball at, then how many seconds (to the nearest tenth of a second) after the ball is snapped should the quarterback throw the ball to his receiver?

32. Simplify the following as a function of m (for $m = 2, 3, \dots$):

$$\sin(\pi/m)\sin(2\pi/m)\sin(3\pi/m)\dots\sin[(m-1)\pi/m]$$

33. Solve for the real number x to the nearest one hundredth:

$$\pi^{\pi^{\pi^{\pi^{\pi^{\dots}}}}} = 3$$

International Interschool!!!

The next five questions are written in foreign languages. Translate them, and put the numeric answer to the question on your answer sheet (in English please).

34. French: Quel est le résultat de sept multiplié par huit multiplié par neuf divisé par quatre?

35. Spanish: ¿Cuál es el producto de veinte por diecisiete entre cuatro?

36. German: Was ist die endumme von siebtzig und zen mal sex?

37. Japanese:

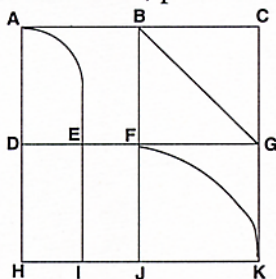
2002 と 2001 を掛けるといくつになりますか。

38. Chinese:

两千零 = 和零的乘积是多少?

39. The road map below will be used to solve the following question. You want to drive from point A to point K in the fastest possible time. The roads all have a set speed limit, given in the chart on the right. The square ACKH is 100 km by 100 km. The roads BJ and DG connect the medians to opposite sides of the square. Road AE is a quarter circle road with radius 25 km attached to a 25 km straight road. Road FK is part of a parabola with K as the vertex. There are certain rules to driving these roads: 1.) You start at point A with 0 speed. 2.) You accelerate from point A at 2 m s^{-2} . 3.) If you are changing direction by more than 45 degrees, then you must decelerate by 1 m s^{-2} 100 meters before the turning point. 4.) Once you are on a new road, you must accelerate at a rate of $\pm 1 \text{ m s}^{-2}$ until you reach the new speed. 5.) As you approach point K you must decelerate at a constant rate starting 200 meters from point K so that your speed at point K is 0.

On your answer sheet, put down the order (by points) of the fastest trip according to the rules above.



Speed Limits

Roads AC, AH, CK, and HK	40 m s^{-1}
Roads BJ and DG	45 m s^{-1}
Roads AI and BG	35 m s^{-1}
Road FK	25 m s^{-1}

40. In question 39, how many minutes (to the nearest minute) does the fastest trip take?

41. A point P lies inside triangle ABC . What is the minimum possible value of the expression:

$$\frac{\overline{PA} \times \overline{PB} \times \overline{PC}}{p_a \times p_b \times p_c},$$

where p_a , p_b , and p_c is the distance from the point P to the side opposite angles A , B , and C , respectively.

42. A lollipop costs 19 cents, a balloon costs 21 cents, and a pack of gum costs 26 cents. Jennifer spends \$1.53 to buy some of these things. Assuming there was no tax, what did she buy and how many of each of these three items did she get?

43. What are the four roots of the following polynomial equation:

$$247x^4 - 1622x^3 + 3728x^2 + 118x - 4199 = 0$$

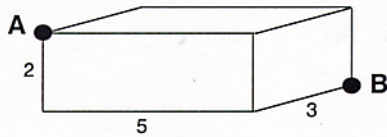
44. What is the product of all the values of x that satisfy $x^{2002} = 2002^{2002}$?

45. If you add up each of the digits in the page numbers of a book that I like to read, the result is 15,333. How many pages are in this book?

46. The portrait below is of a (mostly) self-taught Italian mathematician of the 1500s. Who is it? (Hint: There's a good reason he's wearing a hat)



47. An ant is on the top corner, point A, of a rectangular box with sides of length 2, 3, and 5, as shown in the figure below. What is the shortest distance he can travel to reach point B, the opposite corner?



48. $(5177_8)(\underline{\quad? \quad}_8) = 212100120_6$

49. In the addition problem below, the sum is given as well as the sum of the series of digits that appear in each line. Each dash is to be filled in with one of the digits 1 through 9. All digits will be used once and only once. Note that one digit has been entered as a starting point. On your answer sheet, fill in the rest of the digits.

$$\begin{array}{r}
 _ _ _ _ = 15 \\
 _ _ _ \underline{9} = 19 \\
 + _ _ _ = 11 \\
 \hline
 2 \ 5 \ 1 \ 1
 \end{array}$$

50. The three-dimensional puzzle below is composed of clues that are either simple arithmetic problems or arithmetic concepts. No number begins with a zero, and all answers are whole numbers—no fractions or decimals are involved. Note that you'll be writing the answers in three directions—down to the left (A); down to the right (B); and from the lower left to the right (C). Warning: it is easy to write an answer in the wrong direction, so be aware as you solve.

A CLUES

1. One fourth of 21-A
2. Five times 16-C, times 19-C
3. This number squared is 11-C
5. 1-A times 12-C
7. Nine more than half of 12-C
9. One less than 10-A
10. Twice 16-C
14. 21-A times 16-C
17. Twice 1-A
21. This number squared is 7-B

B CLUES

1. 7-B times 19-C
3. One more than 9-A
4. 3-B minus 1-A
5. Last two digits of 7-A
6. 21-A, reversed
7. The square of a square
8. Three times 13-C
15. A multiple of 5-B
19. Half of 4-B
23. Twice 5-B

C CLUES

11. The square of a square
12. Four times the square of a square
13. The square of a square
16. 17-A plus 5-B
18. Twice 19-C
19. This number squared is 13-C
20. Half of 22-C
22. 1-A plus 19-C
23. 22-C, reversed
24. 12-C times 16-C

