2002 National Mu Alpha Theta Convention
Alpha Division–Ciphering Solutions

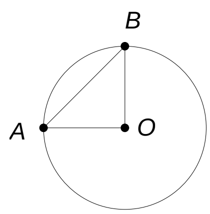
1. **(3)** Since $12^2 = 2^4 \cdot 3^2$ is not divisible by 32 and $12^3 = 2^6 \cdot 3^3$ is, our answer is $n = 3$.

2. **$(-1/13)$** Since $|A^{-1}| = 1/|A|$ and $|A| = (2)(-3) - (1)(7) = -13$, we have $|A^{-1}| = -1/13$.

3. **$(3\sqrt{3})$** Since ADB is inscribed in a semi-circle, $\angle D$ is a right angle. Hence $AD \parallel CE$ and $\triangle ABD \sin \triangle CBE$. Thus, $BE/BC = DB/AB$ so $BE = (3/4)DB$. From right triangle ABD, we find $DB = 4\sqrt{3}$, so $BE = 3\sqrt{3}$.

4. **(145)** All scores are of the form $2n + 5m$, where $0 \le n + m \le 30$. On the low end, only 1 and 3 cannot be achieved (since 4 and 5 can, all other suitably low scores can be reached via $4 + 2k$ and $5 + 2k$). On the high end, suppose we think of the test as starting with 150 points and the student losing 3 points for a question left blank and losing 5 for an incorrect answer. Then, the score is of the form $150 - 3n - 5m$. The quantity $3n + 5m$ cannot take on the values 1,2,4, or 7 (corresponding to scores 149, 148, 146, and 143, respectively). Hence, there are a total of 6 scores out of 151 which cannot be achieved, leaving 145 which can be.

5. **$(27\pi + 18)$** The first inequality is a circle centered at $(3, 4)$ with radius 6. The second inequality gives us a line which intersects the circle at the points $(3, 10)$ and $(-3, 4)$. Thus, the area is three-quarters of the circle plus the triangle AOB shown, or $(3/4)(36\pi) + (1/2)(6)(6) = 27\pi + 18$.



6. **(6)** Since

$$\sum_{i=0}^{\infty} \left(\frac{a}{b}\right)^i = \frac{1}{1 - \frac{a}{b}}$$

if $a < b$ and diverges otherwise, our problem reduces to finding pairs (a, b) such that

$$\begin{aligned} \frac{1}{1 - \frac{a}{b}} &\le 2 \\ \therefore\ 1 &\le 2 - \frac{2a}{b} \\ \therefore\ \frac{a}{b} &\le \frac{1}{2} \end{aligned}$$

The only such pairs with $0 < a < b < 6$ are $(1, 2); (1, 3); (1, 4); (1, 5); (2, 4); (2, 5)$, for a total of 6.

7. **$(\frac{31}{63})$** Let

$$p_1 = P(\text{the coin selected is fair and is heads 7 times in a row})$$

and

$$p_2 = P(\text{the coin selected is not fair and is heads 7 times in a row})$$

Thus,

$$p_1 = \left(\frac{124}{125}\right)\left(\frac{1}{2}\right)^7 \text{ and } p_2 = \left(\frac{1}{125}\right)\left(1\right) = \frac{1}{125}.$$

Since we know that the coin did in fact come up heads nine times in a row, this is a conditional probability problem and our answer is the ratio of the probability that a fair coin is picked and comes up heads 7 times in a row (i.e. p_1) to the probability that either a fair coin is chosen that comes up heads seven times or the unfair coin is chosen (i.e. $p_1 + p_2$) The desired probability then is

$$\frac{p_1}{p_1 + p_2} = \frac{\frac{124}{125\cdot 128}}{\frac{124}{125\cdot 128} + \frac{1}{125}} = \frac{124}{124 + 128} = \frac{31}{63}.$$

8. **(5)** Multiplying both sides by $3(m^2 - n^2)$ gives us $6m = m^2 - n^2$, or $m^2 - 6m - n^2 = 0$. Solving this as a quadratic in m gives

$$m = \frac{6 \pm \sqrt{36 + 4n^2}}{2} = 3 \pm \sqrt{9 + n^2}.$$

Since m and n are integers, the only solutions occur when $9 + n^2$ is a perfect square. Therefore, $n = 0$, 4, or -4 and the 5 solutions are $(6, 0); (8, 4); (-2, 4); (8, -4); (-2 - 4)$ ($(0, 0)$ fails).

9. **(−3/4)** We multiply both sides by $x^3 + 3x^2 - 4x - 12 = (x - 2)(x + 2)(x + 3)$ to yield

$$x + 5 = A(x - 2)(x + 3) + B(x + 2)(x + 3) + C(x + 2)(x - 2).$$

Letting $x = -2$ eliminates the B and C terms on the right, leaving $3 = -4A$, or $A = -3/4$.

10. **(29)** Since $4k^2 \equiv 2 \pmod 7$, we have $2k^2 \equiv 1 \pmod 7$ (because $4k^2 = 7q + 2$ implies $q = 2q'$ for some q' since q must be even, so $2k^2 = 7q' + 1$). We need only check values of k from 0 to 6 since $2(7n + r)^2 \equiv 98n^2 + 28nr + 2r^2 \pmod 7 \equiv 2r^2 \pmod 7$, i.e. $2k^2 \equiv 2(k + 7n)^2 \pmod 7$ for all n. (For example, $2 \cdot 4^2 \equiv 2 \cdot 11^2 \pmod 7 \equiv 2 \cdot 18^2 \pmod 7$, etc.) Testing 0 through 6 yields 2 and 5 as solutions. Thus, all numbers of the form $7n + 2$ or $7n + 5$ are solutions. There are 15 of the former $(2, 9, \ldots, 100 = 7 \cdot 14 + 2)$ and 14 of the latter $(5, 12, \ldots, 96 = 7 \cdot 13 + 5)$, for a total of 29.

11. **(46/3)** Since $\cot^2 x = \csc^2 x - 1$, we have $\cot^2 x = 49/3 - 1 = 46/3$.

12. **(−1)** The sum is $(-1 + 0 + 1 + 2 + \cdots + 8)(1 - n) = 35(1 - n)$. Thus, $35(1 - n) = 70$ and $n = -1$.