2002 Mu- Individual Test Solutions

1. Pick a starting digit. There are 9 choices, then for each successive digit, there are 3 choices (one more, one less, stay the same). $9 \cdot 3^5 = 2187$ $\boxed{B}$

2) Evaluate $\int_0^1 \sqrt{(1-x)(1+x)}dx$.

Note this is the equation for a circle of radius 1, and that this is one quarter of a circle of radius one, so it is ¼*pi. (C).

3) Find the derivative with respect to x of $\sin x \tan x + \cos x - \sec x$.

This is equivalent to (sin(x)^2+cos^2-1)/cos(x)=0/cos(x)=0. (E)

4) Find $\frac{d}{dx}[\cos(2x)]$.

By chain rule this equals -2*sin(2x)=-4sin(x)cos(x). (D)

5) Evaluate $\int x^2 e^{x+1} dx$.
Integrate by parts using u=x^2 and v=e^(x+1) and get:
x^2*e^(x+1)-int(2x*e^(x+1)) repeat and get:
x^2*e^(x+1)-(2x*e^(x+1)-int(2*e^(x+1)))
x^2*e^(x+1)-2x*e^(x+1)+2*e^(x+1)+C
e*e^x*(x-1)^2+e^(x+1)+C (D)

6) Compute the ratio of the area to the perimeter of a triangle with sides measuring 10, 12, and 14.

We use Heron's formula to find the area: s=(10+12+14)/2=18.
A=sqrt(18*(18-10)*(18-12)*(18-14))=sqrt(18*8*6*4)=3*2*2*2sqrt(6)=24sqrt(6)
P=10+12+14=36
A/P=24sqrt(6)/36=2*sqrt(6)/3 (A)

7) The probability of picking 2 of the same:

2 white $= \dfrac{\binom{4}{2}}{\binom{18}{2}}$ 2 Blue $= \dfrac{\binom{6}{2}}{\binom{18}{2}}$ 2 green $= \dfrac{\binom{8}{2}}{\binom{18}{2}}$

$\dfrac{6+15+28}{153} = \dfrac{49}{153}$ $\boxed{A}$

8. Using the chain rule:

-e^x*cos(x)*sin(x)+e^x*cos(x)^2-e^x*sin(x)^2
In calculating the second derivative we can ignore any term which would have a cos(x) in it since this will evaluate to 0:
-e^x*sin(x)^2-e^x*sin(x)^2=-2*e^(pi/2)*sin(pi/2)^2=-2*e^(pi/2) (D)

9. $\lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{x\to 0} 2\cdot\frac{\sin 2x}{2x} = 2\cdot 1 = 2$ $\boxed{D}$

10. $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$ $\boxed{B}$

11. $u = t+2$ $\quad t = u-2$
$du = dt$ $\quad t^2 = (u-2)^2$

$$\int_1^4 \frac{(u-2)^2}{u^{1/2}}\,du = \int \frac{u^2-4u+4}{u^{1/2}}\,du$$
$$= \int \left(u^{3/2} - 4u^{1/2} + 4u^{-1/2}\right) du$$
$$= \left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}4u^{3/2} + 2\cdot 4u^{1/2}\right]_1^4$$
$$= \left(\tfrac{64}{5} - \tfrac{64}{3} + 16\right) - \left(\tfrac{2}{5} - \tfrac{8}{3} + 8\right) = \tfrac{26}{15}$$ (B)

12. Find $\int \sin^3 x\cos x - \sin x\cos^3 x dx$.
int(sinx*cos(x)*(sin(x)^2-cos(x)^2))
u=sin(x)cos(x)
du=sin(x)^2-cos(x)^2
int(u)
u^2/2+C
sin(x)^2*cos(x)^2/2+C (A)

13. Let x=arcos(2).
Then we know that e^(ix)=cos(x)+i*sin(x)
But we know cos(x)=2 and sin(x)=i*sqrt(3)
i*x=ln(2+i*i*sqrt(3))
x=ln(2-sqrt(3))/i
x=ln(1/(2+sqrt(3))/i
x=ln(2+sqrt(3))*i (B)

14) Given that a certain cylinder has a volume V, what is the minimum surface area that the cylinder can have?

V=pi*r^2*h h=V/(pi*r^2)
SA=2*pi*r*h+2*pi*r^2
SA=2*pi*r*V/(pi*r^2)+2*pi*r^2=2*V/r+2*pi*r^2
SA`=-2V/r^2+4*pi*r=0
4*pi*r=2V/r^2
2*pi*r^3=V
r=cuberoot(V/(2*pi))
SA=(2V+2*pi*r^3)/r
SA=(3V)/cuberoot(V/(2*pi))
SA=3*cuberoot(2*pi*V^2) (B)

15) Find the remainder when $x^5+4x^4-3x^2+1$ is divided by x^2-1.

This follows from long division. (E)

16) What is the minimum distance from (2,3) to a point on the line y=-x+8.

The line has slope –1, so a line perpendicular to it would have slope 1. Thus, we solve for an equation of the form y=x+b, which goes through (2,3). This is y=x+1. then we find the intersection of y=-x+8 and y=x+1, which is x+1=-x+8, or 2x=7, x=7/2. Thus the point on the line closest to (2,3) is (7/2, 9/2). Thus, by the distance formula, d=sqrt((3/2)^2+(3/2)^2)=sqrt(9/2)=3*sqrt(2)/2. (E)

17) Find the middle entry of the 100th row of Pascal's Triangle.

Using the formula, that would be entry 50, aka 100!/(50!)^2 (B)

18) Find the sum of the values for the solutions for x, y, and z:

x+2y+6z=10
3x+4y+2z=12
2x+3y+3z=6

Solving this any number of ways yields the solution: x=42, y=-31,z=5, so the sum is 16. (B)

19) How many rectangles can be drawn on an 8x8 chess board which have odd area?

Only rectangles with an odd length and width will have odd area. Given a square on the board we can calculate the number of rectangles with odd area which can be drawn with that square as the upper right hand corner. If we break the board into 2x2 squares we can see that in all these squares this number will be the same, and is, incidentally, the same as the number of rectangles which can be drawn inside a 4x4 square, which is as follows:

16	12	8	4
12	9	6	3
8	6	4	2
4	3	2	1

The sum of the numbers in the above grid is 100, and since each represents 4 squares, the total is 400 (D).

20) Evaluate $\sum_{k=0}^{\infty}\frac{3^{k+1}}{k!}$.

Recall that the Taylor expansion of e^x is sum(x^n/n!,n=0..inf). If we pull a three out of the above we get 3*sum(3^n/n!,n=0..inf), or 3*e^3. (E)

21) Evaluate $\prod_{1}^{200}\left(\frac{n}{n+1}\right)^2$.

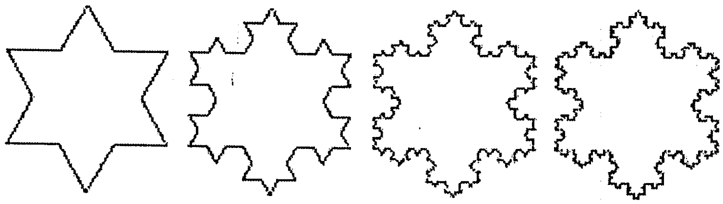
Look at the first several terms:

½*1/2*2/3*2/3*3/4*3/4... Notice how the previous terms drop, so only the very last term will be left, so the answer is (1/201)^2 or 1/40401. (E)

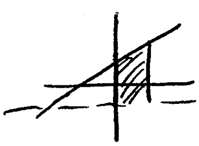
22) Determine the convergence or divergence of $\prod_{1}^{\infty}\frac{n^2+1}{n^2}$.

This is the same as product(1+1/n^2, n=1..inf), which by a theorem converges iff sum(1/n^2,n=1..inf) does, and since it is a p series it does. Thus the product converges. (A)

23) To construct a Koch Snowflake, first you draw an Equilateral Triangle. Then you trisect each of the sides, and construct on the middle segment another equilateral triangle. If this is repeated infinitely, the resulting shape will be a Koch Snowflake (see picture). If a Koch Snowflake is created on an equilateral triangle with sides of length 2, what is it's area?



Recall that the area of an equilateral A=sqrt(3)/4*s^2. Clearly we're looking for an infinite summation for the area, so we write
A=sqrt(3)+sum(3sqrt(3)/4*(4/9)^k,k=1..inf)=sqrt(3)+3sqrt(3)/4*9/5-3sqrt(3)/4=8*sqrt(3)/5 (E)

24. 

$$V = \pi\int r^2 h = \pi\int_0^1 (x+1+1)^2\,dx = \pi\int_0^1 (x+2)^2\,dx =$$

$$\pi\left[\frac{(x+2)^3}{3}\right]_0^1 = \pi\left(9-\frac{8}{3}\right) = \frac{19}{3}\pi \quad \boxed{C}$$

25) Evaluate $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\dots}}}}$.

The key to this problem is noticing that since it repeats, we can express the fraction in terms of itself: x=1/(2+1/(3+x))
2x+x/(x+3)=1
2x^2+6x+x-x-3=0
2x^2+6x-3=0 and by the quadratic formula:
x=(-6(+or-)sqrt(36+24))/4
x=-3/2(+or-)sqrt(60)/4=(sqrt(15)-3)/2 (C)

26. $X = \sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$

$x = \sqrt{2+x}$

$x^2 = 2+x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or -1 A

$x = 2$

27. There are 4^6=4096 possible die combinations. There can be a tie with two die being the same, or a tie with 3 dice being the same, and we can calculate the number of each pattern as follows:
Two Die Tie:
AABBCD=6!/(2!*2!) orders * 4!/(2!*2!) ways to derange the letters assigned in the pattern=1080
AABBCC=6!/(2!*2!*2!) orders * 4 ways to choose which 3 letters to use=360
Three Die Tie:
AAABBB=6!/(3!*3!) orders * 4*3/2 ways to choose 2 letters=120
(1080+360+120)/4096=195/512. A

28. Let $a = \log_{4n} 40\sqrt{3} \rightarrow \begin{cases} (4n)^a = 40\sqrt{3} \\ 4^a n^a = 40\sqrt{3} \end{cases}$

$\therefore a = \log_{3n} 45$ $\qquad n^a = \dfrac{40\sqrt{3}}{4^a}$

$(3n)^a = 45$

$3^a n^a = 45$

$n^a = \dfrac{45}{3^a}$

$$\frac{45}{3^a} = \frac{40\sqrt{3}}{4^a}$$

$$\left(\frac{3}{4}\right)^a = \frac{45}{40\sqrt{3}} = \frac{9}{8\sqrt{3}}$$

$$\frac{3^a}{2^{2a}} = \frac{3^2}{2^3 \cdot 3^{1/2}} = \frac{3^{3/2}}{2^3}$$

$$a = 3/2$$

$$n^{3/2} = \frac{45}{3^{3/2}}$$

$$n^3 = \left(\frac{45}{3^{3/2}}\right)^2 = \frac{45 \cdot 45}{27} = 75 \quad \boxed{E}$$

29. $\cot \frac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

$\cot \frac{1}{2}\left[\arcsin -\frac{3}{5}\right] = \cot \frac{1}{2}\theta$ where $\theta = \arcsin\left(-\frac{3}{5}\right)$

$\therefore \cos\theta = \frac{4}{5}$

$\sqrt{\frac{1+4/5}{1-4/5}} = \sqrt{\frac{9/5}{\frac{1}{5}}} = \sqrt{9} = 3$

But θ in quadrant IV so $\cot\frac{1}{2}\theta = -3$

$\boxed{A}$

30. $\boxed{A}$

31. The difference will be a number of the form $(2n+1)+(2n+3)+\ldots(2n+2k-1)$ with $k>1$
One can establish by induction that this is $k(2n+k)$. Hence 43 is a possible answer.
$44=12^2-10^2$; $45=7^2-2^2$; $48=8^2-4^2$
ans is $\boxed{A}$

32. $d=(13+1)\cdot(5+1)=14(6)=84$ $\boxed{A}$

33. $-\frac{b}{a}=\frac{500}{1000}=\frac{1}{2}$ (A)

34. $\binom{20}{4}=4845$ $\boxed{D}$

35. $-\frac{b}{a}=\frac{1}{1}=1$ (E)

36. $(105311)^2-(105305)^2=(105311+105305)(105311-105305)$
$=6(210616)=1263696$ (A)

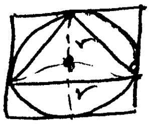
37. $Y=300x+30000-x^2-100x=-x^2+200x+30{,}000$
vertex $=\frac{-b}{2a}=\frac{-200}{-2}=100$ $(100,40000)$ $\boxed{D}$

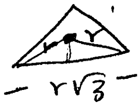
38. $m=\frac{7}{3}$ $y-100=\frac{7}{3}(x-50)$
when $x=0$, $y-100=\frac{7}{3}(-50)=-\frac{350}{3}$ $\boxed{A}$
$y=-\frac{350}{3}+100=-\frac{50}{3}$ $\boxed{C}$

39. $\frac{n(n+1)}{2}=\frac{1000(1001)}{2}=500(1001)=500500$ $\boxed{C}$

40. There are 21 different ways the treasure can be divided $\boxed{B}$

41. Draw 2 evens or draw an odd and a multiple of 4
$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+2\left[\left(\frac{1}{2}\right)\cdot\left(\frac{1}{5}\right)\right]=\frac{1}{4}+\frac{1}{5}=\frac{9}{20}$ $\boxed{C}$

42.  area of square $= (2r)^2 = 4r^2$

area of inscribed $\Delta :=$ 

$$\frac{(r\sqrt{3})^2\sqrt{3}}{4} \qquad \frac{r^2\cdot 3\cdot\sqrt{3}}{4} = \qquad \frac{3r^2\sqrt{3}}{4}$$

$$\text{ratio} = \frac{\frac{3\cancel{r^2}\sqrt{3}}{4}}{4\cancel{r^2}} = \frac{3}{16}\sqrt{3} \quad \boxed{B}$$

43. $\underline{9}\cdot\underline{10}\cdot\underline{10}\cdot\underline{1}\cdot\underline{1} = 900$ $\boxed{D}$

44. Since $\{a_n\}$ is a bounded and montonic function, it converges to L. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

Thus $L = \frac{1}{3-L} \Rightarrow 3L - L^2 = 1$ $\qquad L^2 - 3L + 1 = 0$

$L = \frac{3\pm\sqrt{5}}{2}$ $\qquad$ since $a_n \le 2$ $\quad L = \frac{3-\sqrt{5}}{2}$

$\boxed{C}$

45. The numbers can't be more than 10 places long.
Thus we have $9 + 9^2 + 9^2\cdot 8 + 9^2\cdot 8\cdot 7 + 9^2\cdot 8\cdot 7\cdot 6 + \dots 9^2\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 8,877,690$ (D)

46. $100 + 2\left[\frac{1}{2}(100) + \left(\frac{1}{2}\right)^2(100) + \left(\frac{1}{2}\right)^3(100) + \dots\right] = 300$ $\boxed{B}$

$100 + 2\left[\frac{50}{1-\frac{1}{2}}\right] = 100 + \frac{100}{1/2} = 100 + 200 = 300$

47. When determinant $= 0$.

$$\left|\begin{matrix} 1 & 4 & C \\ 2 & -1 & 7 \\ 3 & -2 & 11 \end{matrix}\right| \begin{matrix} 1 & 4 \\ 2 & -1 \\ 3 & -2 \end{matrix}$$

Determinant $= -11 + 84 + -4C + 3C + 14 - 88 =$

$0 = -1 - C$

$C = -1$ (B)

48. $x = \sqrt{2+\sqrt{2}} - \sqrt{2-\sqrt{2}}$

$x^2 = 2+\cancel{\sqrt{2}} - 2\sqrt{2} + 2 - \cancel{\sqrt{2}}$

$x^2 = 4 - 2\sqrt{2}$

$x^4 = 16 - 16\sqrt{2} + 8 = 24 - 16\sqrt{2}$

$x^8 = 576 - 768\sqrt{2} + 512 = 1088 - 768\sqrt{2}$

$384x^2 - x^8 = 384(4-2\sqrt{2}) - 1088 + 768\sqrt{2}$

$= 448$ C

49. minimum at vertex

vertex: $x = \frac{-c}{2a} = \frac{-c}{2}$

$f\left(-\frac{c}{2a}\right) = \frac{c^2}{4} - \frac{c^2}{2} + 2 = -1$

$c^2 - 2c^2 + 8 = -4$

$-c^2 + 12 = 0$

$12 = c^2$

$\pm 2\sqrt{3} = c$

c must be positive $(2\sqrt{3})$ E

50. $\log_3 \frac{45}{7} = \log_3 45 - \log_3 7$

$\log_3 9 + \log_3 5 - \log_3 7 =$ D

$2 + x - y$