Alpha-
matrices Topic Test
Solutions

b) 1. $\begin{bmatrix} 2 & -8 \\ 6 & -7 \end{bmatrix} = \begin{bmatrix} 2x-3 & 3y+1 \\ -5z+1 & -2w-5 \end{bmatrix}$

$2x-3=2$ $\quad$ $3y+1=-8$ $\quad$ $-5z+1=6$ $\quad$ $-2w-5=-7$

$2x=5$ $\quad$ $3y=-9$ $\quad$ $-5z=5$ $\quad$ $-2w=-2$

$x=2.5$ $\quad$ $y=-3$ $\quad$ $z=-1$ $\quad$ $w=1$

d 2. $\begin{bmatrix} 3x+y & x-2y & 2x \\ 2x-5y & 3x-2y & x+y \end{bmatrix} + \begin{bmatrix} 2x & -3y & 5x+y \\ 3x+2y & x-4y & 2x \end{bmatrix} =$

$\begin{bmatrix} 5x+y & x-5y & 7x+y \\ 5x-3y & 4x-6y & 3x+y \end{bmatrix}$

a 3. $A = \begin{bmatrix} 3 & -11 & -1 \\ -6 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -2 & 8 \\ 2 & -4 & -5 \end{bmatrix}$

$2A = \begin{bmatrix} 6 & -22 & -2 \\ -12 & 8 & 18 \end{bmatrix}$ and $3B = \begin{bmatrix} 21 & -6 & 24 \\ 6 & -12 & -15 \end{bmatrix}$

$2A-3B = \begin{bmatrix} 6 & -22 & -2 \\ -12 & 8 & 18 \end{bmatrix} - \begin{bmatrix} 21 & -6 & 24 \\ 6 & -12 & -15 \end{bmatrix}$

$= \begin{bmatrix} -15 & -16 & -26 \\ -18 & 20 & 33 \end{bmatrix}$

(e) 4. $3A + \begin{bmatrix} 3 & -2 \\ 4 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 13 \\ -8 & 2 \\ 1 & -1 \end{bmatrix}$

$$3A = \begin{bmatrix} 6 & 13 \\ -8 & 2 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 4 & -7 \\ -2 & 5 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 15 \\ -12 & 9 \\ 3 & -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 \\ -4 & 3 \\ 1 & -2 \end{bmatrix}$$

5. $$4\begin{bmatrix} x & y \\ z & -1 \end{bmatrix} = 2\begin{bmatrix} y & z \\ -x & 1 \end{bmatrix} + 2\begin{bmatrix} 4 & x \\ 5 & -x \end{bmatrix}$$

$$\begin{bmatrix} 4x & 4y \\ 4z & -4 \end{bmatrix} = \begin{bmatrix} 2y & 2z \\ -2x & 2 \end{bmatrix} + \begin{bmatrix} 8 & 2x \\ 10 & -2x \end{bmatrix}$$

$$\begin{bmatrix} 4x & 4y \\ 4z & -4 \end{bmatrix} = \begin{bmatrix} 2y+8 & 2z+2x \\ -2x+10 & 2-2x \end{bmatrix}$$

$2-2x=-4$

$-2x=-6$

$x=3$

$-2x+10=4z$

$-2(3)+10=4z$

$-6+10=4z$

$4=4z$

$1=z$

$2z+2x=4y$

$2(1)+2(3)=4y$

$2+6=4y$

$8=4y$

$2=y$

6. $\begin{bmatrix} 2 & x & 3 \\ -1 & 2 & 7 \\ z & 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 3 & -3 & -1 \\ 2 & 0 & y \end{bmatrix} =$

(d)

$$\begin{bmatrix} 12+3x & 2-3x & 10-x+3y \\ 17 & -7 & -7+7y \\ 3z+11 & z-15 & 5z-5-2y \end{bmatrix}$$

7. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$

(d)

trace of $A = 1 - 2 + 3 = 2$

8. $A = \begin{bmatrix} 1 & 2 \\ x & -1 \end{bmatrix}$

(e)

$$A^3 = \begin{bmatrix} 1 & 2 \\ x & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ x & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ x & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2x & 0 \\ 0 & 2x+1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ x & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2x & 2+4x \\ 2x^2+x & -2x-1 \end{bmatrix}$$

(b) 9. $\begin{bmatrix} 2 & -3 \\ x & 1 \end{bmatrix}^{-1} = \frac{1}{2+3x} \begin{bmatrix} 1 & 3 \\ -x & 2 \end{bmatrix}$

(c) 10. $\begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 9-2x & 0 \\ 0 & -2x+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$9-2x = 1$$

$$-2x = -8$$

$$x = 4$$

(d) 11. $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\frac{1}{2} A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

$(A^{-1})^{-1} = A = \frac{1}{-8} \begin{bmatrix} 8 & -4 \\ -6 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0.5 \\ 0.75 & -0.25 \end{bmatrix}$

(a) 12. $\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} -1.5 & -1 \\ -1.25 & -2 \end{bmatrix}$

13. $\begin{vmatrix} x & 3x \\ -2 & x \end{vmatrix} = 0$

(b)

$x^2 + 6x = 0$

$x(x+6) = 0$

$x = -6, 0$

14. $\begin{vmatrix} x-3 & 2 \\ 4 & x-1 \end{vmatrix} = (x-3)(x-1) - 8$

(b)

$= x^2 - 4x + 3 - 8$

$= x^2 - 4x - 5$

15. $\begin{vmatrix} x & y & 1 \\ 2 & 3 & -1 \\ -2 & -1 & 3 \end{vmatrix} = x\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - y\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -2 & -1 \end{vmatrix}$

(a)

$= x(8) - y(4) + 4$

$= 8x - 4y + 4$

16. $\begin{vmatrix} 2 & -3 \\ x & 4 \end{vmatrix} = \begin{vmatrix} 5 & 4 \\ -1 & 3 \end{vmatrix}$

(a) $8 + 3x = 19$

$3x = 11$

$x = \frac{11}{3}$

17. Theorem: If A is an $n \times n$ matrix and c is a scalar, then $|cA| = c^n |A|$.

(b) $|A| = -3$

$|2A| = 2^4 |A| = 2^4(-3) = 16(-3) = -48$

18. Theorem: If A is invertible, then $|A^{-1}| = \frac{1}{|A|}$.

(b) $|A| = 3+2x$

$|A^{-1}| = \frac{1}{3+2x}$

19. Theorem: If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(a)

$$A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$$

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ &= \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 35 & 17 \\ 4 & 10 \end{bmatrix} \end{aligned}$$

20. Theorem: If A is an $n \times n$ invertible matrix, then $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$.

d

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

21. $$\text{Area}_\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

b

$\pm$ is chosen to give a positive area

$$= \pm\frac{1}{2}\begin{vmatrix} -1 & 2 & 1 \\ 2 & 2 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(6)$$

$$= 3$$

22. $3\vec{u} = 6i - 9j + 3k$

$2\vec{v} = -6i + 10j + 8k$

(d) $\vec{w} = i - 2j + 3k$

$3\vec{u} - 2\vec{v} + \vec{w} = 6i - 9j + 3k + 6i - 10j - 8k$

$+ \; i - 2j + 3k$

$= 13i - 21j - 2k$

23. $\|\vec{v}\| = \sqrt{3^2 + (-2)^2 + (\sqrt{7})^2}$

$= \sqrt{9 + 4 + 7}$

(a) $= \sqrt{20}$

$= 2\sqrt{5}$

24. unit vector in the direction of $\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

(c) $\|\vec{v}\| = \sqrt{5^2 + 3^2 + (-6)^2}$

$= \sqrt{25 + 9 + 36}$

$= \sqrt{70}$

$\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{5}{\sqrt{70}} i + \dfrac{3}{\sqrt{70}} j - \dfrac{6}{\sqrt{70}} k$

25. $\vec{u}\cdot\vec{v} = 2(-3) + (-4)(2) + (1)(2)$

(d) $= -6 - 8 + 2$

$= -12$

26. $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

$\vec{u}-\vec{v} = 2i - 3j + 2k + 5i + 4j - 6k$

$= 7i + j - 4k$

(d) $\|\vec{u}-\vec{v}\| = \sqrt{7^2 + 1^2 + (-4)^2}$

$= \sqrt{49 + 1 + 16}$

$= \sqrt{66}$

27. $c\vec{v} = 2ci - cj + 3ck$

$\|c\vec{v}\| = \sqrt{(2c)^2 + (-c)^2 + (3c)^2}$

(c) $= \sqrt{4c^2 + c^2 + 9c^2}$

$= \sqrt{14c^2}$

$\sqrt{14c^2} = 1$

$14c^2 = 1$

$c = \pm \frac{1}{\sqrt{14}}$

28. $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$, $0 \le \theta \le \pi$

$\vec{u}\cdot\vec{v} = 6(2) + (-3)(2) + (2)(-5)$

$= 12 - 6 - 10$

$= -4$

$\|\vec{v}\| = \sqrt{2^2+2^2+(-5)^2}$

$= \sqrt{4+4+25}$

b

$= \sqrt{33}$

$\|\vec{u}\| = \sqrt{6^2+(-3)^2+2^2}$

$= \sqrt{36+9+4}$

$= \sqrt{49}$

$= 7$

$\cos\theta = \dfrac{-4}{7\cdot\sqrt{33}}$

$\theta \approx 95.7^\circ$

29) $\vec{v} \times \vec{u} = \begin{vmatrix} i & j & k \\ -2 & 2 & -3 \\ 6 & -3 & 2 \end{vmatrix}$

d

$$= i \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} - j \begin{vmatrix} -2 & -3 \\ 6 & 2 \end{vmatrix} + k \begin{vmatrix} -2 & 2 \\ 6 & -3 \end{vmatrix}$$

$$= -5i - 14j - 6k$$

30) $\text{Area}_{\square} = \|\vec{u} \times \vec{v}\|$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

b

$$= i \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} - j \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} + k \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 8i - 10j + 4k$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{8^2 + (-10)^2 + 4^2}$$

$$= \sqrt{64 + 100 + 16}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5}$$

14

Bonus: $|\lambda I - A| = \begin{vmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & 10 \\ 0 & 0 & \lambda-4 \end{vmatrix}$

$$= \lambda \begin{vmatrix} \lambda-4 & 10 \\ 0 & \lambda-4 \end{vmatrix} - 3 \begin{vmatrix} 4 & 10 \\ 0 & \lambda-4 \end{vmatrix} - 5 \begin{vmatrix} 4 & \lambda-4 \\ 0 & 0 \end{vmatrix}$$

2

$$= \lambda(\lambda-4)(\lambda-4) - 3(4)(\lambda-4)$$

$$= \lambda(\lambda^2 - 8\lambda + 16) - 12\lambda + 48$$

$$= \lambda^3 - 8\lambda^2 + 16\lambda - 12\lambda + 48$$

$$= \lambda^3 - 8\lambda^2 + 4\lambda + 48$$

$$\lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$$

$$\lambda = -2, 4, 6$$