Alpha Number Theory Test

1) $(\cos^2 1° + \cos^2 89°) + (\cos^2 2° + \cos^2 88°) + \ldots + \cos^2 45° + \cos^2 90°$

(a) since $\cos x = \sin(90 - x) \Rightarrow \cos 89° = \sin(90-89) = \sin 1°$

so ...

$(\cos^2 1° + \sin^2 1°) + (\cos^2 2° + \sin^2 2°) + \ldots + \cos^2 45° + \cos^2 90° \Rightarrow$

$44(1) + \frac{1}{2} + 0 = 44.5$

(b) 2) $1^1 + 3^2 + 5^3 = 1 + 9 + 125 = 135$

(b) 3) 36 has factors 1, 2, 3, 4, 6, 9, 12, 18, + 36

(c) 4) $3 \Rightarrow 3, 9, 7, 1, \ldots$ $\quad \frac{2002}{4} = 500$ w/remainder 2

$2 \Rightarrow 2, 4, 6, 8, \ldots$ so 2^{nd} for $3 \Rightarrow 9$; $2 \Rightarrow 4$ $\quad 9 - 4 = 5$

(b) 5) The only way to get some of 4 different digits to $= 10$ is $1+2+3+4 = 10$. Figure 4 digits can be arranged $4!$ or 24 ways.

(c) 6) $5^{5^5} = x \quad 5^{3125} = x \quad 3125 \log 5 = \log x$

$2184.28 = \log x \quad 10^{2185} \approx x$

\# of digits

(b) 7) 6 is LCD for 2 and 3 and $2 \cdot 3 = 6$

(c) 8) $63_8 = 6(8) + 3 = 51 \quad 110100_2 = 2^5 + 2^4 + 2^2 = 52$

both of which are less than 56

(d) 9) Find LCD for 6, 8 + 9 $\Rightarrow 2 \cdot 3; 2^3; 3^2 \Rightarrow 2^3 \cdot 3^2 = 72$

(c) 10) $\sqrt{19^2 + 19^2 + \ldots + 19^2} = \sqrt{19^4} = 19^2 = 361$

(a) 11) since odd + odd = even 2 must be a factor + 2 is smallest prime number

Alpha Number Theory pg. 2 Answer Key

(a) 12) $4+2\sqrt{3} = (1+\sqrt{3})^2$ $28+10\sqrt{3} = (5+\sqrt{3})^2$

so $\dfrac{1+\sqrt{3}-(5+\sqrt{3})}{15} = \dfrac{-4}{15}$

(d) 13) $1^6 = 1 \to 1$ $4^6 = 4096 \to 19 \to 10 \to 1$

$2^6 = 64 \to 10 \to 1$ $5^6 = 15625 \to 19 \to 10 \to 1$

$3^6 = 729 \to 18 \to 9$

pattern 1, 1, 9, 1, 1, 9

(a) 14) taking the differences: 247, 901, 682 + 907

247 is smallest

(d) 15) $1000 = 2^3 \cdot 5^3 \to \{0,1,2,3\}$

$\{0,1,2,3\}$ $4 \cdot 4 = 16$ factors

(b) 16) Since the numbers are distinct, this eliminates any solutions.

(a) 17) girls = 18 + boys boys + girls = 44 ⇒ boys = 13, girls = 31

Since each team must have at least 1 girl + 1 boy, the max numbers of teams can only be 13.

(a) 18) $]n[\, = n^2 + 38$ and $\boxed{n} = 2n+3$ so possibilities would be -8 or 2. Only choice of the two is 2.

(d) 19) The only choice not divisible by another factor is 127

(e) 20) The only way to get a whole number is to use $\dfrac{3}{1} - \dfrac{4}{2} = 1$ which makes $A + C = 3 + 4 = 7$

(a) 21) Square both sides and combine equations you get $9 + 16 + 24(\sin A \cos B + \sin B \cos A) = 37$

$24(\sin(A+B)) = 12$

$\sin(A+B) = \frac{1}{2}$

$\sin C = \sin(180 - (A+B)) = \frac{1}{2}$

$C = 30°$

Alpha Number Theory pg 3 ANSWER Key

(b) 22) $x+2 = \frac{1}{x-2}$ $x^2-4=1$ $x^2-5=0$ $x = \pm\sqrt{5}$

(c) 23) $R=0$ since $R+E=E \Rightarrow O=9$ & $U+V=N>10 \Rightarrow N=2F+1$
Since N is odd, $U+V=15 \Rightarrow U$ or $V = 7$ or 8; $U+V=13$ with
$U, V = 5$ or 8, or 7 or 6. The only way to get all odds
for N, I, E is to use $U+V=15$, $N=5$.

(e) 24) $5 \cdot 13 = 65$ which is not divisible by 4

(b) 25) a total of $3! = 6$ different numbers begin
with each 3, 4, 5, 6 so the 13th has to be the
smallest beginning with 5.

(d) 26) If x = odd then x^2-29 = even and $>2 \Rightarrow$ Not prime
test x = even
$6^2-29=\underline{7}$; $8^2-29=35$; $10^2-29=\underline{71}$; $12^2-29=115$; $14^2-29=\underline{167}$
3 primes out of 9 choices $\Rightarrow \frac{1}{3}$

(a) 27) $(2-1)+(3-1)+\ldots+(10-1)$
$1+2+3+\ldots+9 = 45$

(d) 28) $2^{1999} \cdot 5^{1999} \cdot 5^2 = \underbrace{10^{1999}}_{\text{zeros}}(25)$ so $2+5=7$

(a) 29) a number 1 less than a multiple of 5 will end in
4 or 9; 4 cannot be 1 greater than a multiple
of 4 so consider $10d+9$. Only 9, 29, 49, 69 + 89 are
1 more than multiple of 4, and only 29 and 89 are
prime. Sum $29+89 = 118$

Alpha Number Theory pg 4 Answer Key

30) prime factorization of multiples of 30 will look like $2^i 5^k 3^j N$, i, j, k are positive and $N \geq 1$

(c)

The number of divisors $= (i+1)(j+1)(k+1)n$

Since we want 36 divisors, & $j \geq 1, k \geq 1$, then $i \in \{1,2,3,5,8\}$

Now test each possibility; i.e. $i=1 \Rightarrow 2 \cdot 3^2 \cdot 5^2 \cdot 7 = 3150$

$2^2 \cdot 3^5 \cdot 5 = 4860$ or $2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ smallest

31) We need a nonzero digit so the multiple of 5 has to be a multiple of 125. Test cases

(b)

$121 \cdot 122 \cdot 123 \cdot 124 \cdot 125 = 28{,}143{,}753{,}000$. Sum of digits $1+2+1 = 4$

32)

(b)

$7 + e + f = 18$
$\Rightarrow e + f = 11$

$\underbrace{e+f}_{11} + g = 18$
$\Rightarrow g = 7$ (left of x)

$7 + x + i = 18$
$7 + x + 8 = 18$
$x = 3$

$j + k + 8 = 18$
$\Rightarrow j + k = 10$
$i + j + k = 18$
so $i = 8$

33) if divisible by 7 the difference is divisible by 7

(c)

$ABC = 100A + 10B + C$
$CBA = A + 10B + 100C$
$99(A-C)$ since 99 is not divisible by 7

then $A-C$ must be

$A=7, C=0 \Rightarrow$ no 3 digit #
$A=8, C=1 \Rightarrow 861$ and 168 ✓
$A=9, C=2 \Rightarrow 952, 259$

34) Counting the number of odds forms a pattern

(b)

1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16 . . .

100^{th} row will be 101^{st} term in this sequence

The sequence "self-doubles"

Theorem by Kummer states number of odd numbers in Pascal's $\Delta = 2^r$ where $r = \#$ of 1's in binary expansion $n = 100 = 1100100_2 \Rightarrow 1$ occurs 3 times

So $2^3 = 8$

(d) 35) $\underbrace{4({}_5C_2)}_{\text{horizontal pair points}} + \underbrace{5({}_4C_2)}_{\text{vertical}} + \underbrace{2}_{\text{5 units apart}} = 72$ $\qquad \frac{72}{190} = \frac{36}{95}$

${}_{20}C_2$ represents total # of pairs $= 190$